

The super spanning connectivity of arrangement graphs

Pingshan Li Min Xu*

Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing, 100875, China

Abstract

A k -container $C(u, v)$ of a graph G is a set of k internally disjoint paths between u and v . A k -container $C(u, v)$ of G is a k^* -container if it is a spanning subgraph of G . A graph G is k^* -connected if there exists a k^* -container between any two different vertices of G . A k -regular graph G is super spanning connected if G is i^* -container for all $1 \leq i \leq k$. In this paper, we prove that the arrangement graph $A_{n,k}$ is super spanning connected if $n \geq 4$ and $n - k \geq 2$.

Key words: Hamiltonian; Hamiltonian connected; k^* -connected; Arrangement graphs.

1 Introduction

In the field of parallel and distributed systems, interconnection networks are an important research area. Typically, the topology of a network can be represented as a graph in which the vertices represent processors and the edges represent communication links.

For graph definitions and notations, we follow [4]. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices of G . A path P of length k from x to y is a finite sequence of distinct vertices $\langle v_0, v_1, \dots, v_k \rangle$, such that $x = v_0, y = v_k$, and $(v_i, v_{i+1}) \in E$ for $0 \leq i \leq k - 1$. We also represent path P as $\langle v_0, v_1, \dots, v_i, Q, v_j, v_{j+1}, \dots, v_k \rangle$, where Q is the path $\langle v_i, v_{i+1}, \dots, v_j \rangle$. In particular, if $i = j$, we can still represent the path as $\langle v_0, v_1, \dots, v_i, Q, v_i, v_{j+1}, \dots, v_k \rangle$.

A spanning subgraph of G is a subgraph with vertex set $V(G)$. A Hamiltonian graph is a graph with a spanning cycle. A graph is Hamiltonian connected if there exists a spanning path joining any two different vertices.

A k -container $C(u, v)$ of G is a set of k internally disjoint paths between u and v , and the connectivity of G , $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. It follows from Menger's Theorem [15] that there is a k -container between any two distinct vertices of G if G is k -connected.

A k -container $C(u, v)$ of G is a k^* -container if it is a spanning subgraph of G . A graph is k^* -connected if there exists a k^* -container between any two distinct vertices. By this definition, the concept of Hamiltonian connected is the same as 1^* -connected and the concept of Hamiltonian is the same as 2^* -connected. Thus, the concept of k^* -connected is a hybrid concept of connectivity and Hamiltonicity. The study of k^* -connected graphs is motivated by the globally 3^* -connected graphs proposed by M. Albert et al. [3].

The star graph (S_n for short), which was proposed by Akers et al. [2], is a well known interconnection network. The arrangement graph [6], denoted by $A_{n,k}$, refers to a generalized version of S_n . Further, $A_{n,n-1}$ is isomorphic to the n -dimensional star graph S_n [1] and $A_{n,1}$ is isomorphic to the complete graph K_n . The arrangement graph preserves many attractive properties of S_n such as the hierarchical structure, vertex and edge symmetry, simple and optimal routing, and many fault tolerance properties [6]. Some basic properties of $A_{n,k}$ such as average distance [5], Hamiltonicity [8], and embedding [7, 17] have recently been computed or derived.

A graph G is super spanning connected if it is k^* -connected for all $1 \leq k \leq \kappa(G)$. There are many desirable results about super spanning connected of some interconnection networks such as recursive circulant graphs [18], pancake graphs [11], hypercube-like network [13], (n, k) -star graphs [10], k -ary n -cubes [16] and multi-dimensional tori [12]. Since $A_{n,n-1} \cong S_n$ is a bipartite graph with the same number of vertices in each partite set, there is no Hamiltonian path joining any two different vertices in the same part. Hence, $A_{n,n-1}$ is not super spanning connected if $n > 3$. Therefore, we

*Corresponding author.

E-mail address: xum@mail.bnu.edu.cn (M. Xu).

consider an arrangement graph with $n - k \geq 2$. In this paper, we aim to prove that arrangement graphs are super spanning connected if $n \geq 4$ and $n - k \geq 2$.

The rest of this paper is organized as follows. In Section 2, we introduce arrangement graphs and discuss some of their properties. In Section 3, we prove that arrangement graphs are super spanning connected for $n \geq 4$ and $n - k \geq 2$.

2 Arrangement graphs

Throughout this paper, we assume that n and k are positive integers with $n > k$. We use $\langle n \rangle$ to denote the set $\{1, 2, \dots, n\}$. The arrangement graph $A_{n,k}$ is a graph that has the vertex set $V(A_{n,k}) = \{u = u_1 u_2 \dots u_k \mid u_i \in \langle n \rangle, u_i \neq u_j \text{ if } i \neq j\}$ and the edge set $E(A_{n,k}) = \{(p, q) \mid p, q \in V(A_{n,k}) \text{ and } p, q \text{ differ in exactly one position}\}$. From the definition, we know that $A_{n,k}$ is a regular graph of degree $k(n - k)$ with $\frac{n!}{(n-k)!}$ vertices. Figure 1 illustrates the arrangement graph $A_{4,2}$.

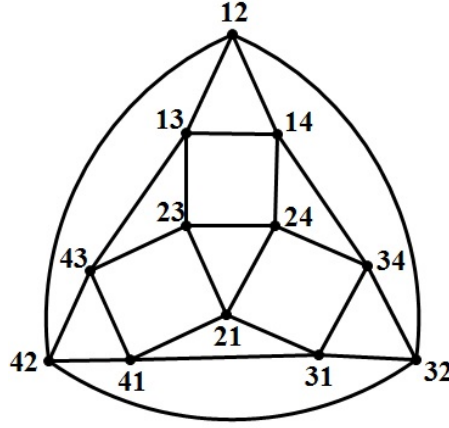


Figure 1: The arrangement graph $A_{4,2}$

Let $u = u_1 u_2 \dots u_k \in V(A_{n,k})$. We denote $(u)_i = u_i$ as the i th coordinate of u for $1 \leq i \leq k$. Let v be a neighbor of u , we denote v as $u^{s(u_i, x)}$ if $v = u_1 u_2 \dots u_{i-1} x u_{i+1} \dots u_k$ for $x \in \langle n \rangle \setminus \{(u)_i : i = 1, 2, \dots, k\}$.

For $i, j \in \langle n \rangle, l \in \langle k \rangle$ and $i \neq j$, suppose that $A_{n,k}^{(l,i)}$ denotes the subgraph of $A_{n,k}$ that is induced by $V(A_{n,k}^{(l,i)}) = \{p \mid p = p_1 p_2 \dots p_k \text{ and } p_l = i\}$. Obviously, $\{V(A_{n,k}^{(l,i)}) \mid 1 \leq i \leq n\}$ forms a partition of $V(A_{n,k})$ and each $A_{n,k}^{(l,i)}$ is isomorphic to $A_{n-1,k-1}$. As a result, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. We use $E^{l=i,j}$ to denote the set of edges between $A_{n,k}^{(l,i)}$ and $A_{n,k}^{(l,j)}$; accordingly, $E^{l=i,j} = \frac{(n-2)!}{(n-k-1)!}$. We also use $A_{n,k}^{(l,I)}$ to denote the subgraph of $A_{n,k}$ that is induced by $\cup_{i \in I} V(A_{n,k}^{(l,i)})$.

Following are some known properties about arrangement graphs.

Lemma 2.1 ([9]) *The arrangement graph $A_{n,k}$ is $(k(n-k)-2)$ -fault-tolerant Hamiltonian, and $(k(n-k)-3)$ -fault-tolerant Hamiltonian-connected for $n-k \geq 2$.*

Lemma 2.2 ([14]) *The arrangement graph $A_{n,k}$ is $(k(n-k)-2)$ -edge-fault-tolerant Hamiltonian connected if not all faulty edges are adjacent to the same vertex.*

In the following, we discuss some properties that will be used in the proof of the main results.

Lemma 2.3 *$A_{4,2}$ is super spanning connected.*

Proof: By Lemma 2.1, $A_{4,2}$ is 1^* -connected and 2^* -connected. We need to construct a 3^* -container and a 4^* -container joining any two different vertices u and v of $A_{4,2}$. Since $A_{4,2}$ is vertex and edge transitive, without loss of generality, we can assume that $u = 12$ and $v = 13, 34, 23, 21$. We list such 3^* -containers as follows:

$\langle 12, 13 \rangle$	$\langle 12, 14, 13 \rangle$	$\langle 12, 42, 43, 41, 21, 31, 32, 34, 24, 23, 13 \rangle$
$\langle 12, 14, 34 \rangle$	$\langle 12, 42, 32, 34 \rangle$	$\langle 12, 13, 43, 23, 24, 21, 41, 31, 34 \rangle$
$\langle 12, 13, 43, 23 \rangle$	$\langle 12, 14, 34, 24, 23 \rangle$	$\langle 12, 42, 32, 31, 41, 21, 23 \rangle$
$\langle 12, 13, 43, 23, 21 \rangle$	$\langle 12, 14, 34, 24, 21 \rangle$	$\langle 12, 42, 32, 31, 41, 21 \rangle$

and such 4*-containers as follows:

$\langle 12, 13 \rangle$	$\langle 12, 14, 13 \rangle$
$\langle 12, 42, 43, 13 \rangle$	$\langle 12, 32, 34, 31, 41, 21, 24, 23, 13 \rangle$
$\langle 12, 13, 43, 23, 24, 34 \rangle$	$\langle 12, 14, 34 \rangle$
$\langle 12, 32, 34 \rangle$	$\langle 12, 42, 41, 21, 31, 34 \rangle$
$\langle 12, 13, 23 \rangle$	$\langle 12, 14, 24, 23 \rangle$
$\langle 12, 32, 34, 31, 21, 23 \rangle$	$\langle 12, 42, 41, 43, 23 \rangle$
$\langle 12, 13, 43, 23, 21 \rangle$	$\langle 12, 14, 34, 24, 21 \rangle$
$\langle 12, 32, 31, 21 \rangle$	$\langle 12, 42, 41, 21 \rangle$

□

Lemma 2.4 Suppose that $n \geq 5, k \geq 2$ and $n - k \geq 2$. Let $I = \{i_1, i_2, \dots, i_m\} \subseteq \langle n \rangle$, then $A_{n,k}^{(i,I)}$ is Hamiltonian connected where $1 \leq i \leq k$.

Proof: Let u and v be any two distinct vertices of $A_{n,k}^{(i,I)}$, we will prove that there exists a Hamiltonian path P of $A_{n,k}^{(i,I)}$ joining u and v . If $|I| = 1$, by Lemma 2.1, it is true. Therefore, we assume that $|I| \geq 2$ in the next proof.

Case 1: $(u)_i \neq (v)_i$.

Without loss of generality, let $(u)_i = i_1$ and $(v)_i = i_m$. There exists at least 3 edges between $A_{n,k}^{(i,i_j)}$ and $A_{n,k}^{(i,i_{j+1})}$ for all $1 \leq j \leq m-1$ owing to $|E^{i=i_j, i_{j+1}}| = \frac{(n-2)!}{(n-k-1)!} \geq 3$. An edge $(x^{i_j}, y^{i_{j+1}}) \in E^{i=i_j, i_{j+1}}$ is chosen such that $(x^{i_j})_i = i_j$, $(y^{i_{j+1}})_i = i_{j+1}$ and $x^1 \neq u, y^m \neq v, x^{i_j} \neq y^{i_j}$ for all $j = 1, 2, \dots, m-1$. Let $y^1 = u, x^m = v$. Since $A_{n,k}$ is Hamiltonian connected, there exists a Hamiltonian path $\langle y^{i_j}, P_j, x^{i_j} \rangle$ of $A_{n,k}^{(i,i_j)}$ joining y^{i_j} to x^{i_j} for $1 \leq j \leq m$. Hence, there exists a Hamiltonian path $P = \langle y^1, P_1, x^1, y^2, P_2, x^2, \dots, y^m, P_m, x^m \rangle$ of $A_{n,k}^{(i,I)}$ joining u to v . See figure 2 for illustration.

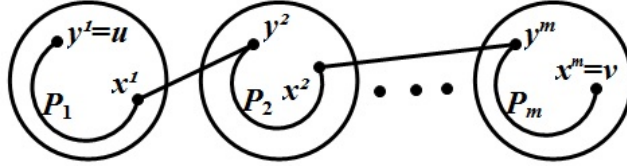


Figure 2: Illustration for case 1 of Lemma 2.4

Case 2: $(u)_i = (v)_i$.

Without loss of generality, let $(u)_i = (v)_i = i_1$. A vertex $x^1 \in V(A_{n,k}^{(i,i_1)}) \setminus \{u, v\}$ is chosen such that $i_2 \notin \{(x^1)_j : j = 1, 2, \dots, k\}$. Obviously, $|\{x' \mid (x^1, x') \in E(A_{n,k}^{(i,i_1)}) \text{ and } i_2 \notin \{(x')_j : 1 \leq j \leq k\}\}| = (n-k-1)(k-1) \geq 2$ owing to $n \geq 5$. Hence, there exists at least two neighbors y^1, z^1 of x^1 such that $i_2 \notin \{(y^1)_j : 1 \leq j \leq k\} \cup \{(z^1)_j : 1 \leq j \leq k\}$. Let $e = (x^1, a) \in E(A_{n,k}^{(i,i_1)})$ and $a \notin \{y^1, z^1\}$. By Lemma 2.2, there exists a Hamiltonian path P_1 of $A_{n,k}^{(i,i_1)} - \bigcup_{x \in N_{A_{n,k}^{(i,i_1)}}(x^1)} \{(x^1, x)\} + \{(x^1, y^1), (x^1, z^1), e\}$

between u and v . Note that the degree of x^1 in P_1 is two, then $\{(x^1, y^1), (x^1, z^1)\} \cap E(P_1) \geq 1$. Without loss of generality, let $(x^1, y^1) \in E(P_1)$, and we can represent P_1 as $\langle u, R_1, x^1, y^1, H_1, v \rangle$. Let $u^j = (x^{j-1})^{s(i_{j-1}, i_j)}$ and $v^j = (y^{j-1})^{s(i_{j-1}, i_j)}$ for $2 \leq j \leq m$. Similarly, there exists a Hamiltonian path P_j of $A_{n,k}^{(i,i_j)}$ such that $(x^j, y^j) \in E(P_j)$ and $i_{j+1} \notin \{(x^j)_l : 1 \leq l \leq k\} \cup \{(y^j)_l : 1 \leq l \leq k\}$ for all $2 \leq j \leq m-1$. We can represent P_j as $\langle u^j, R_j, x^j, y^j, H_j, v \rangle$ for $1 \leq j \leq m-1$. By Lemma 2.1, there exists a Hamiltonian path P_m of $A_{n,k}^{(i,i_m)}$ between u^m and v^m . Hence, there exists a Hamiltonian path $P = \langle u, R_1, x^1, u^2, R_2, x^2, \dots, u^m, P_m, v^m, \dots, y^2, H_2, v^2, y^1, H_1, v \rangle$ of $A_{n,k}^{(i,I)}$ joining u to v . See figure 3 for illustration.

□

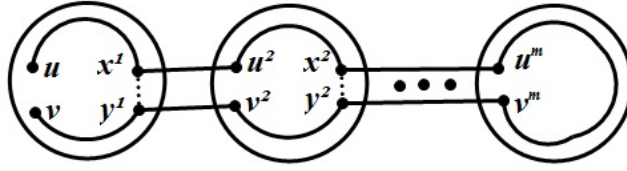


Figure 3: Illustration for case 2 of Lemma 2.4

Lemma 2.5 For $m \in \langle n \rangle$. Suppose that $A = \{u^1, u^2, \dots, u^m\}, B = \{v^1, v^2, \dots, v^m\}, A \cap B = \emptyset, A, B \subseteq V(A_{n,k})$. If there exists a number $t \in \langle k \rangle$ such that $(u^i)_t \neq (u^j)_t, (v^i)_t \neq (v^j)_t$ for $1 \leq i \neq j \leq m$, then there exists m disjoint paths H_1, H_2, \dots, H_m from A to B such that $V(\cup_{j=1}^m H_j) = V(A_{n,k})$.

Proof: We partite $A_{n,k}$ to $\cup_{i=1}^n A_{n,k}^{(t,i)}$. Suppose that $|\{(u^i)_t : 1 \leq i \leq m\} \cap \{(v^i)_t : 1 \leq i \leq m\}| = l$. Without loss of generality, we can assume that $(u^i)_t = (v^i)_t$ for $1 \leq i \leq l$. By Lemma 2.4, there exists a Hamiltonian path H_i of $A_{n,k}^{(t,(u^i)_t)}$ joining u^i to v^i for $1 \leq i \leq l$ and a Hamiltonian path H_j of $A_{n,k}^{(t,\{(u^j)_t, (v^j)_t\})}$ joining u^j to v^j for $l+1 \leq j \leq m-1$. Let $I = \langle n \rangle \setminus (\{(u^i)_t : 1 \leq i \leq m-1\} \cup \{(v^i)_t : 1 \leq i \leq m-1\})$. By Lemma 2.4, there exists a Hamiltonian path H_m of $A_{n,k}^{(i,I)}$ joining u^m to v^m . Obviously, H_1, H_2, \dots, H_m form the desired paths. See figure 4 for illustration. \square

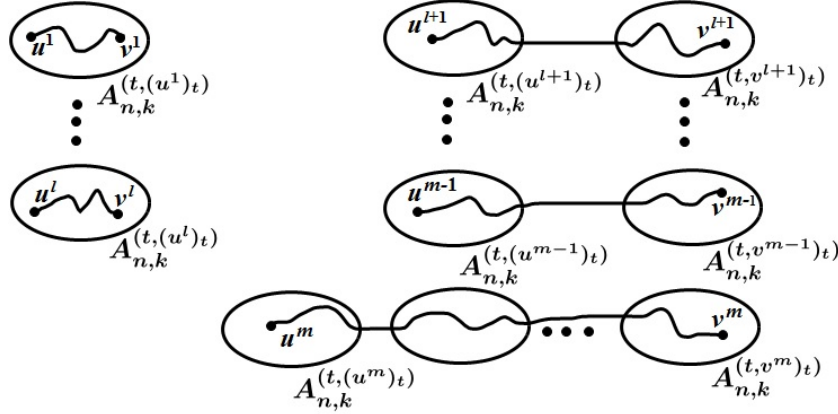


Figure 4: Illustration for Lemma 2.5

3 The super spanning connectivity of arrangement graphs

Lemma 3.1 Suppose that $n \geq 4, n-k \geq 2$, then $A_{n,k}$ is l^* -connected for $(n-k)(k-1)+1 \leq l \leq (n-k)k$.

Proof. We prove the Lemma by induction.

Basis step: Since $A_{n,1}$ is isomorphic to the complete graph K_n and by Lemma 2.3, $A_{4,2}$ is super spanning connected. Thus, the result holds for $A_{n,1}$ and $A_{4,2}$.

Induction step: Suppose that $A_{n-1,k-1}$ is $(n-k)(k-1)^*$ -connected.

We need to find an l^* -container between any two different vertices u and v of $A_{n,k}$ with $k \geq 2$ for $(n-k)(k-1)+1 \leq l \leq (n-k)k$. We use U to denote the set $\{(u)_i : 1 \leq i \leq k\}$ and V to denote the set $\{(v)_i : 1 \leq i \leq k\}$.

Case 1: $\{i \mid (u)_i = (v)_i : 1 \leq i \leq k\} \neq \emptyset$.

Without loss of generality, let $(u)_k = (v)_k = \alpha$. Suppose that:

$$\begin{aligned} U \cap V &= \{x_1, x_2, \dots, x_t, \alpha\}, \\ U \setminus (U \cap V) &= \{u'_{t+1}, u'_{t+2}, \dots, u'_{k-1}\}, \\ V \setminus (U \cap V) &= \{v'_{t+1}, v'_{t+2}, \dots, v'_{k-1}\}, \\ \langle n \rangle \setminus (U \cup V) &= \{w_1, w_2, \dots, w_{n+t-2k+1}\}. \end{aligned}$$

We partite $A_{n,k}$ to $\cup_{i=1}^n A_{n,k}^{(k,i)}$. By induction, there exists an $(n-k)(k-1)^*$ -container $\{P_1, P_2, \dots, P_{(n-k)(k-1)}\}$ of $A_{n,k}^{(k,\alpha)}$ joining u and v . By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k,w_i)}$ joining $u^{s(\alpha,w_i)}$ to $v^{s(\alpha,w_i)}$ for $1 \leq i \leq n+t-2k+1$ and a Hamiltonian path H_j of $A_{n,k}^{(k,v'_j)} \cup A_{n,k}^{(k,u'_j)}$ joining $u^{s(\alpha,v'_j)}$ to $v^{s(\alpha,u'_j)}$ for $t+1 \leq j \leq k-1$.

(a) $(n-k)(k-1)+1 \leq l \leq (n-k)(k-1)+n+t-2k+1$. (If $n+t-2k+1=0$, then (a) does not occur.)

Let $l = (n-k)(k-1)+l'$, then $1 \leq l' \leq n+t-2k+1$. We set $P_{(n-k)(k-1)+i} = \langle u, u^{s(\alpha,w_i)}, R_i, v^{s(\alpha,w_i)}, v \rangle$ for $1 \leq i \leq l'-1$. By Lemma 2.4, there exists a Hamiltonian path H of $A_{n,k}^{(k,I)}$ joining $u^{s(\alpha,w_{l'})}$ to $v^{s(\alpha,w_{l'})}$ where $I = \langle n \rangle - \{\alpha, w_1, w_2, \dots, w_{l'-1}\}$. We set $P_l = \langle u, u^{s(\alpha,w_{l'})}, H, v^{s(\alpha,w_{l'})}, v \rangle$. Obviously, $\{P_1, P_2, \dots, P_l\}$ forms an l^* -container of $A_{n,k}$ joining u to v . See figure 5 for illustration.

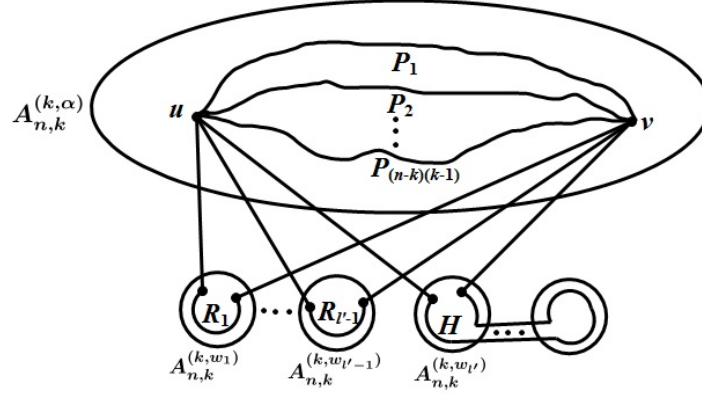


Figure 5: Illustration for case 1 (a) of Lemma 3.1

(b) $(n-k)(k-1)+n+t-2k+2 \leq l \leq (n-k)k$. (If $k-t=1$, then (b) does not occur.)

Let $l = (n-k)(k-1)+n+t-2k+1+l'$, then $1 \leq l' \leq k-t-1$. We set

$$P_{(n-k)(k-1)+i} = \langle u, u^{s(\alpha,w_i)}, R_i, v^{s(\alpha,w_i)}, v \rangle \text{ for } 1 \leq i \leq n+t-2k+1,$$

$$P_{(n-k)(k-1)+n-2k+1+j} = \langle u, u^{s(\alpha,v'_j)}, H_j, v^{s(\alpha,u'_j)}, v \rangle \text{ for } t+1 \leq j \leq t+l'-1.$$

Let $I = \langle n \rangle \setminus \{\alpha, w_1, w_2, \dots, w_{n+t-2k+1}, u'_{t+1}, \dots, u'_{t+l'-1}, v'_{t+1}, \dots, v'_{t+l'-1}\}$. By Lemma 2.4, there exists a Hamiltonian H' of $A_{n,k}^{(k,I)}$ joining $u^{s(\alpha,v'_{t+l'})}$ to $v^{s(\alpha,u'_{t+l'})}$. We set $P_l = \langle u, u^{s(\alpha,v'_{t+l'})}, H', v^{s(\alpha,u'_{t+l'})}, v \rangle$. Obviously, $\{P_1, P_2, \dots, P_l\}$ forms an l^* -container of $A_{n,k}$ joining u to v . See figure 6 for illustration.

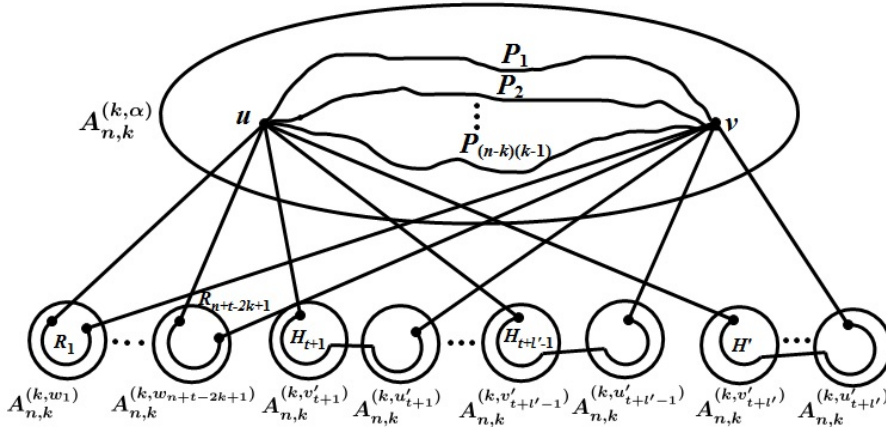


Figure 6: Illustration for case 1 (b) of Lemma 3.1

Case 2: $\{i \mid (u)_i = (v)_i : 1 \leq i \leq k\} = \emptyset$.

Case 2.1 : $\{i \mid (u)_i \in V, (v)_i \in U\} \neq \emptyset$.

Without loss of generality, we can assume that $(u)_k \in V, (v)_k \in U$. Let $(u)_k = \alpha, (v)_k = \beta$. Suppose that:

$$\begin{aligned}
U \cap V &= \{x_1, \dots, x_t, \alpha, \beta\}, \\
U \setminus (U \cap V) &= \{u'_{t+1}, u'_{t+2}, \dots, u'_{k-2}\}, \\
V \setminus (U \cap V) &= \{v'_{t+1}, v'_{t+2}, \dots, v'_{k-2}\}, \\
\langle n \rangle \setminus (U \cup V) &= \{w_1, w_2, \dots, w_{n+t-2k+2}\}.
\end{aligned}$$

Since $n - k \geq 2$, there exists a element $\gamma \in \langle n \rangle \setminus V$. Set $y = x_1 \cdots x_t v'_{t+1} \cdots v'_{k-2} \gamma \alpha$ and $z = x_1 \cdots x_t v'_{t+1} \cdots v'_{k-2} \gamma \beta$. Thus, $u \neq y, z \neq v$. Let $S = \langle n \rangle \setminus \{(y)_i : 1 \leq i \leq k\} \cup \{\beta\} = \langle n \rangle \setminus \{(z)_i : 1 \leq i \leq k\} \cup \{\alpha\} = \{s_1, s_2, \dots, s_{n-k-1}\}$.

By induction, there exists an $(n-k)(k-1)^*$ -container $\{P_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}^{(k,\alpha)}$ joining u to y , and an $(n-k)(k-1)^*$ -container $\{Q_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}^{(k,\beta)}$ joining z to v . We can represent P_{ij} as $\langle u, P'_{ij}, y^{ij}, y \rangle$, Q_{ij} as $\langle z, z^{ij}, Q'_{ij}, v \rangle$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n-k$ where

$$\begin{aligned}
y^{ij} &= \begin{cases} y^{s((y)_i, s_j)} : 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ y^{s((y)_i, \beta)} : 1 \leq i \leq k-1, j = n-k, \end{cases} \\
z^{ij} &= \begin{cases} z^{s((y)_i, s_j)} : 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ z^{s((y)_i, \alpha)} : 1 \leq i \leq k-1, j = n-k. \end{cases}
\end{aligned}$$

Obviously, $(y^{ij}, z^{ij}) \in E(A_{n,k})$ when $1 \leq i \leq k-1$ and $1 \leq j \leq n-k-1$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k,(y)_i)}$ joining $(y^{i(n-k)})^{s(\alpha, (y)_i)}$ to $(z^{i(n-k)})^{s(\beta, (z)_i)}$ for $1 \leq i \leq k-2$. As a result, there exists $(n-k)(k-1)$ internally disjoint paths $\{M_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}$ joining u to v such that

$$V\left(\bigcup_{i=1}^{k-1} \bigcup_{j=1}^{n-k} M_{ij}\right) = V(A_{n,k}^{(k, \{x_1, \dots, x_t, v'_{t+1}, \dots, v'_{k-2}, \alpha, \beta\})})$$

where

$$M^{ij} = \begin{cases} \langle u, P'_{ij}, y^{ij}, z^{ij}, Q'_{ij}, v \rangle \text{ for } 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(\alpha, (y)_i)}, R_i, (z^{ij})^{s(\beta, (z)_i)}, z^{ij}, Q'_{ij}, v \rangle \text{ for } 1 \leq i \leq k-2, j = n-k, \\ \langle u, P'_{ij}, y^{ij}, y, z, z^{ij}, Q'_{ij}, v \rangle \text{ for } i = k-1, j = n-k. \end{cases}$$

See figure 7 for illustration.

(a) $(n-k)(k-1) + 1 \leq l \leq (n-k)(k-1) + n + t - 2k + 2$. (If $n + t - 2k + 2 = 0$, then (a) does not occur.)

Let $l = (n-k)(k-1) + l'$, then $1 \leq l' \leq n + t - 2k + 2$. By Lemma 2.4, there exists a Hamiltonian path R'_i of $A_{n,k}^{(k, w_i)}$ joining $u^{s(\alpha, w_i)}$ to $v^{s(\beta, w_i)}$ for $1 \leq i \leq l' - 1$. Let $I = \{u'_{t+1}, \dots, u'_{k-2}, w_{l'}, \dots, w_{n+t-2k+2}\}$, by Lemma 2.4, there exists a Hamiltonian path $R'_{l'}$ of $A_{n,k}^{(k, I)}$ joining $u^{s(\alpha, w_{l'})}$ to $v^{s(\beta, w_{l'})}$. Set $M_{ki} = \langle u, u^{s(\alpha, w_i)}, R'_i, v^{s(\beta, w_i)}, v \rangle$ for $1 \leq i \leq l'$. To construct the l^* -container, we only need to combine figure 7 and l' paths $M_{k1}, M_{k2}, \dots, M_{kl'}$ in figure 8.

(b) $(n-k)(k-1) + n + t - 2k + 3 \leq l \leq (n-k)k$. (If $k - t = 2$, then (b) is not happened.)

Let $l = (n-k)(k-1) + n + t - 2k + 2 + l'$, then $1 \leq l' \leq k - t - 2$. In this case, we have $U \setminus (U \cap V) \neq \emptyset$. Note that $\gamma \in \langle n \rangle \setminus V$. Without loss of generality, we can assure that $\gamma = u'_{k-2}$.

To construct the l^* -container, we need to

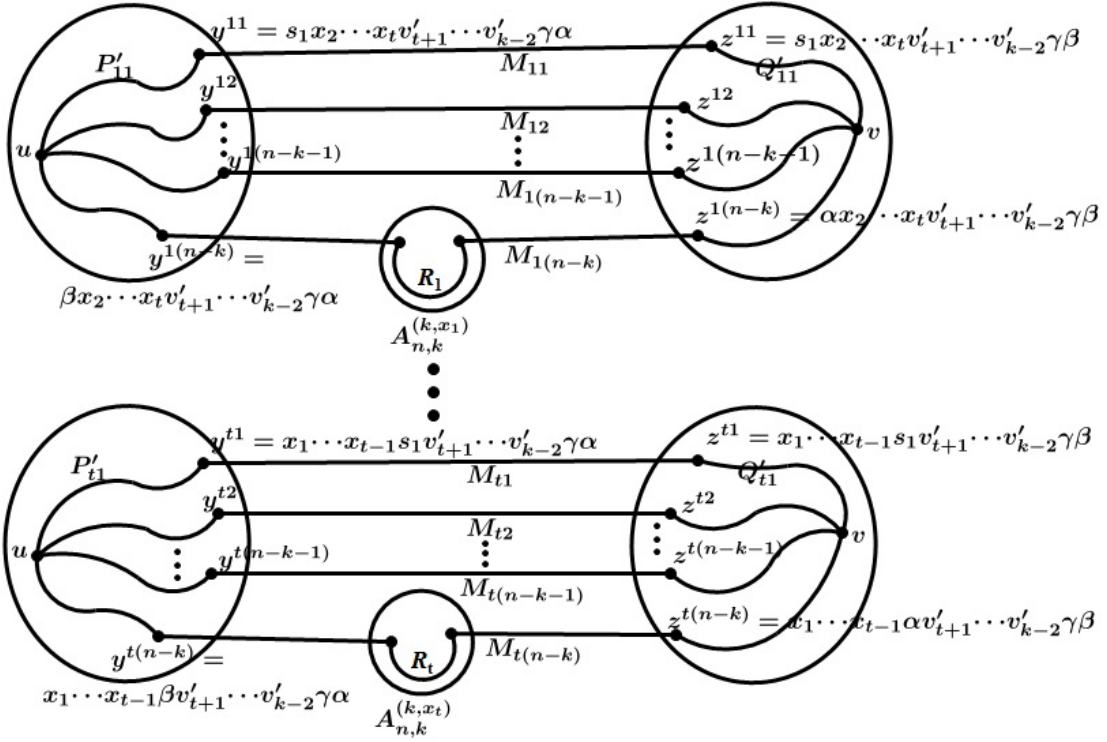
Step 1: By Lemma 2.4, there exists a Hamiltonian path R''_i of $A_{n,k}^{(k, w_i)}$ joining $u^{s(\alpha, w_i)}$ to $v^{s(\beta, w_i)}$ for $1 \leq i \leq n + t - 2k + 2$.

Combine figure 7 and $(n + t - 2k + 2)$ paths $M'_{k1}, M'_{k2}, \dots, M'_{k(n+t-2k+2)}$ in figure 9 where

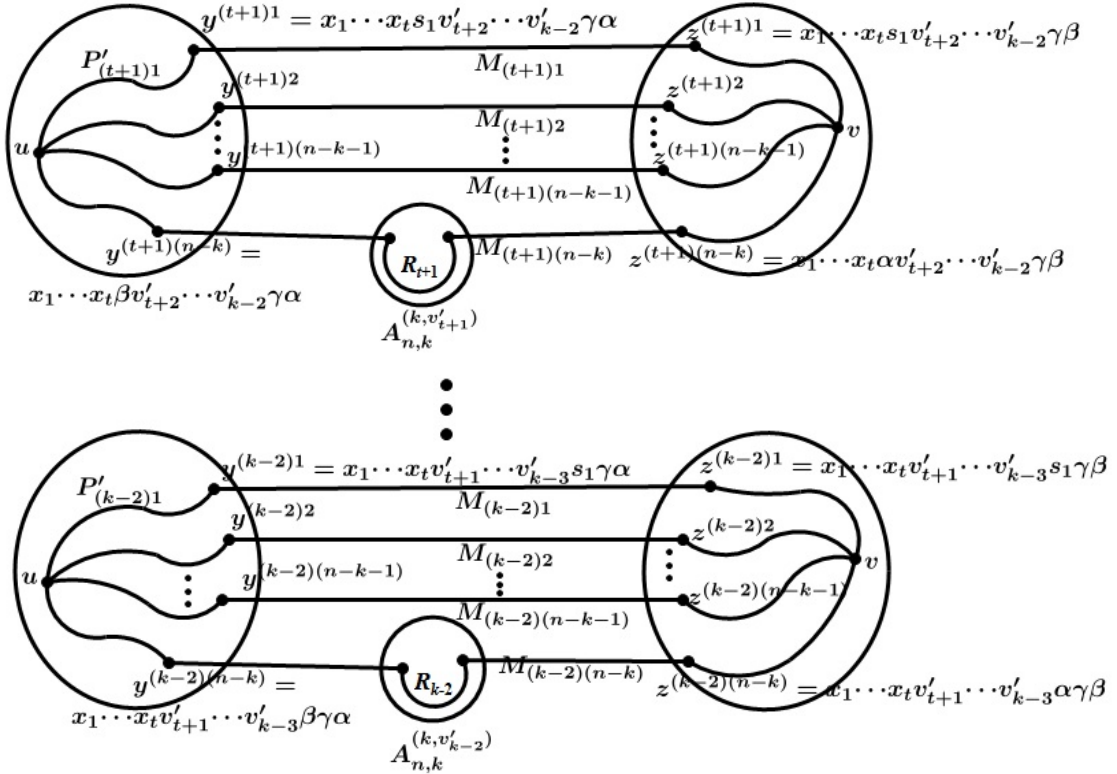
$$M'_{ki} = \langle u, u^{s(\alpha, w_i)}, R''_i, v^{s(\beta, w_i)}, v \rangle \text{ for } 1 \leq i \leq n + t - 2k + 2.$$

Step 2: By Lemma 2.4, there exists a Hamiltonian path H_j of $A_{n,k}^{(k, u'_{t+j})}$ joining $(y^{(t+j)(n-k)})^{s(\alpha, u'_{t+j})}$ to $v^{s(\beta, u'_{t+j})}$ and a Hamiltonian path H'_j of $A_{n,k}^{(k, v'_{t+j})}$ joining $u^{s(\alpha, v'_{t+j})}$ to $(z^{(t+j)(n-k)})^{s(\beta, v'_{t+j})}$ for $1 \leq j \leq l' - 1$.

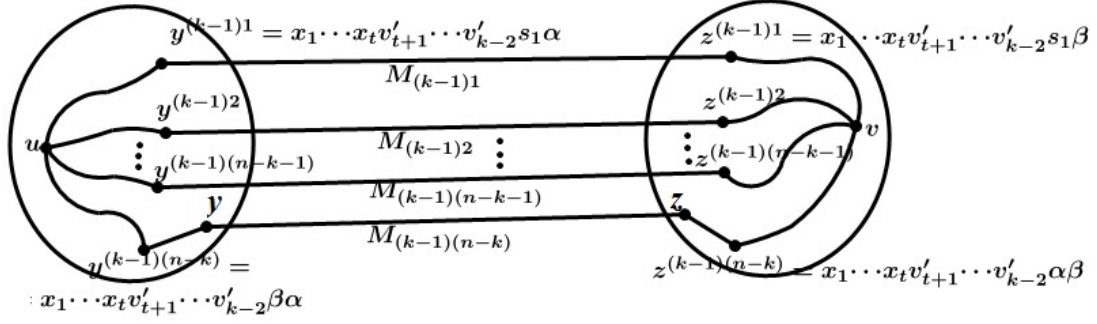
Replace paths $M_{(t+1)(n-k)}, M_{(t+2)(n-k)}, \dots, M_{(t+l'-1)(n-k)}$ in part II of figure 7 by $M'_{(t+1)(n-k)}, M'_{(t+2)(n-k)}, \dots, M'_{(t+l'-1)(n-k)}$ and $M_1, M_2, \dots, M_{l'-1}$ as shown in figure 10 where



Part I



Part II



----- **Part III** -----

Figure 7: The $(n-k)(k-1)$ internally disjoint paths of $A_{n,k}$ of case 2.1, 2.2 of Lemma 3.1

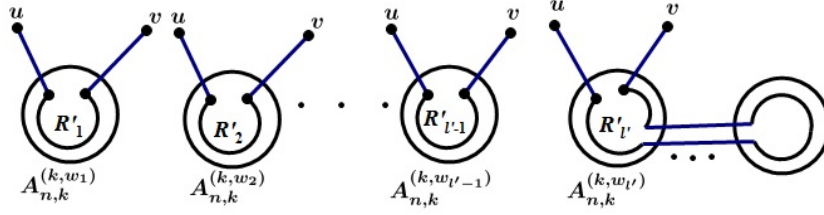


Figure 8: The paths $M_{k1}, \dots, M_{kl'}$ of case 2.1(a) in Lemma 3.1

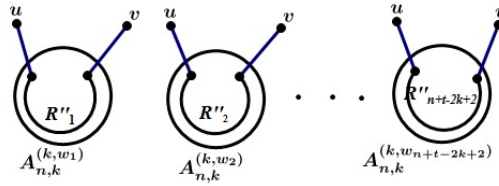


Figure 9: The paths $M_{k1}, \dots, M_{k(n+t-2k+2)}$ of case 2.1(b) in Lemma 3.1

$$\begin{aligned}
M'_{(t+j)(n-k)} &= \langle u, P'_{(t+j)(n-k)}, y^{(t+j)(n-k)}, (y^{(t+j)(n-k)})^{s(\alpha, u'_{t+j})}, H_j, v^{s(\beta, u'_{t+j})}, v \rangle, \\
M_j &= \langle u, u^{s(\alpha, v'_{t+j})}, H'_j, (z^{(t+j)(n-k)})^{s(\beta, v'_{t+j})}, z^{(t+j)(n-k)}, Q'_{(t+j)(n-k)}, v \rangle \quad \text{for } 1 \leq j \leq l' - 1.
\end{aligned}$$

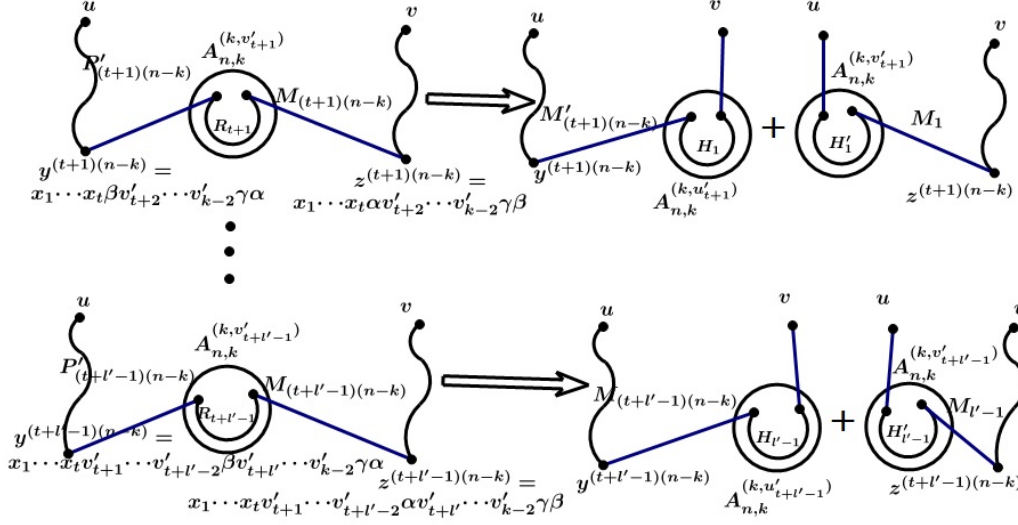


Figure 10: Illustration for step 2 of case 2.1(b) in Lemma 3.1

Step 3: Let $I = \{u'_{t+l'}, u'_{t+l'+1}, \dots, u'_{k-2}\}$. By Lemma 2.4, there exists a Hamiltonian path H of $A_{n,k}^{(k, u'_{k-2})}$ joining $u^{s(\alpha, v'_{k-2})}$ to $(z^{(k-2)(n-k)})^{s(\beta, v'_{k-2})}$ and a Hamiltonian path R of $A_{n,k}^{(k, I)}$ joining $(y^{(k-1)(n-k)})^{s(\alpha, u'_{k-2})}$ to $v^{s(\beta, u'_{k-2})}$.

Replace $M_{(k-2)(n-k)}$ and $M_{(k-1)(n-k)}$ in part II and part III of figure 7 by $M'_{(k-2)(n-k)}$, $M'_{(k-1)(n-k)}$ and $M_{l'}$ as shown in figure 11 where

$$\begin{aligned}
M'_{(k-2)(n-k)} &= \langle u, u^{s(\alpha, v'_{k-2})}, H, (z^{(k-2)(n-k)})^{s(\beta, v'_{k-2})}, z^{(k-2)(n-k)}, Q'_{(k-2)(n-k)}, v \rangle, \\
M'_{(k-1)(n-k)} &= \langle u, P'_{(k-1)(n-k)}, y^{(k-1)(n-k)}, (y^{(k-1)(n-k)})^{s(\alpha, u'_{k-2})}, R, v^{s(\beta, u'_{k-2})}, v \rangle, \\
M_{l'} &= \langle u, P'_{(k-2)(n-k)}, y^{(k-2)(n-k)}, y, z, z^{(k-1)(n-k)}, Q'_{(k-1)(n-k)}, v \rangle.
\end{aligned}$$

Case 2.2 : $\{i \mid (u)_i \notin V, (v)_i \in U\} \neq \emptyset$ or $\{(i \mid (v)_i \notin U, (u)_i \in V) \neq \emptyset\}$.

Without loss of generality, let $(u)_k \notin V, (v)_k \in U$, and let $(u)_k = \alpha, (v)_k = \beta$. Suppose that

$$\begin{aligned}
U \cap V &= \{x_1, x_2, \dots, x_t, \beta\}, \\
U \setminus (U \cap V) &= \{u'_{t+1}, u'_{t+2}, \dots, u'_{k-2}, \alpha\}, \\
V \setminus (U \cap V) &= \{v'_{t+1}, v'_{t+2}, \dots, v'_{k-1}\}, \\
\langle n \rangle \setminus (U \cup V) &= \{w_1, w_2, \dots, w_{n+t-2k+1}\}.
\end{aligned}$$

Without loss of generality, let $v = x_1 x_2 \dots x_t v'_{t+1} \dots v'_{k-1} \beta$. Since $n - k \geq 2$, there exists an element $\gamma \in \langle n \rangle \setminus V$. Set $y = x_1 \dots x_t v'_{t+1} \dots v'_{k-2} \gamma \alpha$ and $z = x_1 \dots x_t v'_{t+1} \dots v'_{k-2} \gamma \beta$. Thus, $u \neq y, z \neq v$. Let $S = \langle n \rangle \setminus \{(y)_i : 1 \leq i \leq k\} \cup \{\beta\} = \langle n \rangle \setminus \{(z)_i : 1 \leq i \leq k\} \cup \{\alpha\} = \{s_1, s_2, \dots, s_{n-k-1}\}$. By induction, there exists an $(n-k)(k-1)^*$ -container $\{P_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}^{(k, \alpha)}$ joining u to y and an $(n-k)(k-1)^*$ -container $\{Q_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}^{(k, \beta)}$ joining z to v . We can represent P_{ij} as $\langle u, P'_{ij}, y^{ij}, y \rangle$, Q_{ij} as $\langle z, z^{ij}, Q'_{ij}, v \rangle$ for $1 \leq i \leq k-1$ and $1 \leq j \leq n-k$ where

$$y^{ij} = \begin{cases} y^{s((y)_i, s_j)} : 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ y^{s((y)_i, \beta)} : 1 \leq i \leq k-1, j = n-k, \end{cases}$$

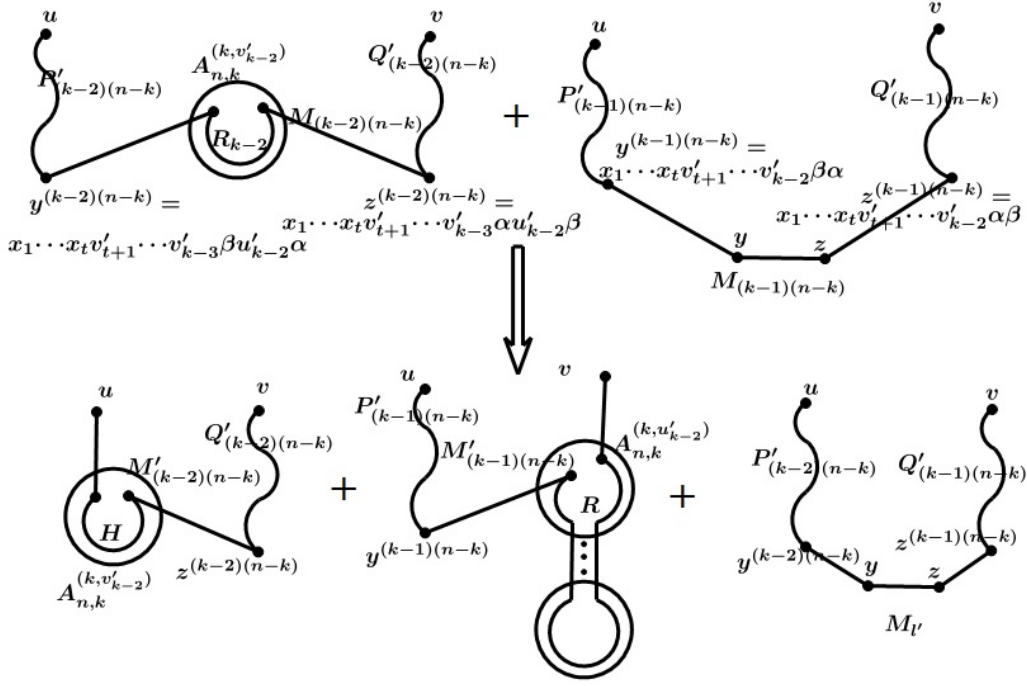


Figure 11: Illustration for step 3 of case 2.1(b) in Lemma 3.1

$$z^{ij} = \begin{cases} z^{s((y)_i, s_j)} : 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ z^{s((y)_i, \alpha)} : 1 \leq i \leq k-1, j = n-k. \end{cases}$$

Obviously, $(y^{ij}, z^{ij}) \in E(A_{n,k})$ when $1 \leq i \leq k-1, 1 \leq j \leq n-k-1$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k, (y)_i)}$ joining $(y^{i(n-k)})^{s(\alpha, (y)_i)}$ to $(z^{i(n-k)})^{s(\beta, (z)_i)}$ for $1 \leq i \leq k-2$. Then, there exists $(n-k)(k-1)$ internally disjoint paths $\{M_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-k\}$ of $A_{n,k}$ joining u to v such that

$$V\left(\bigcup_{i=1}^{k-1} \bigcup_{j=1}^{n-k} M_{ij}\right) = V(A_{n,k}^{(k, \{\alpha, \beta, x_1, \dots, x_t, v'_{t+1}, \dots, v'_{k-2}\})}).$$

where

$$M^{ij} = \begin{cases} \langle u, P'_{ij}, y^{ij}, z^{ij}, Q'_{ij}, v \rangle \text{ for } 1 \leq i \leq k-1, 1 \leq j \leq n-k-1, \\ \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(\alpha, (y)_i)}, R_i, (z^{ij})^{s(\beta, (z)_i)}, z^{ij}, Q'_{ij}, v \rangle \text{ for } 1 \leq i \leq k-2, j = n-k, \\ \langle u, P'_{ij}, y^{ij}, y, z, z^{ij}, Q'_{ij}, v \rangle \text{ for } i = k-1, j = n-k. \end{cases}$$

See figure 7 for illustration.

(a) $(n-k)(k-1) + 1 \leq l \leq (n-k)(k-1) + n + t - 2k + 2$.

Let $l = (n-k)(k-1) + l'$, then $1 \leq l' \leq n + t - 2k + 2$. To construct the l^* -container, we need

Step 1: By Lemma 2.4, there exists a Hamiltonian path R'_i of $A_{n,k}^{(k, w_i)}$ joining $u^{s(\alpha, w_i)}$ to $v^{s(\beta, w_i)}$ for $1 \leq i \leq l' - 1$.

Combine figure 7 and $(l' - 1)$ disjoint paths $M_{k1}, M_{k2}, \dots, M_{kl'}$ in figure 12 where $M_{ki} = \langle u, u^{s(\alpha, w_i)}, R'_i, v^{s(\beta, w_i)}, v \rangle$ for $1 \leq i \leq l' - 1$.

Step 2: Let $I = \langle n \rangle \setminus \{\alpha, \beta, x_1, \dots, x_t, v'_{t+1}, \dots, v'_{k-2}\}$. Then, there exists a Hamiltonian path $R'_{l'}$ of $A_{n,k}^{(k, I)}$ joining $u^{s(\alpha, v'_{k-1})}$ to $(z^{(k-1)(n-k)})^{s(\beta, v'_{k-1})}$.

Replace $M_{(k-1)(n-k)}$ in part III of figure 7 by $M'_{(k-1)(n-k)}$ and $M_{kl'}$ as shown in 13 where

$$M_{kl'} = \langle u, u^{s(\alpha, v'_{k-1})}, R'_{l'}, (z^{(k-1)(n-k)})^{s(\beta, v'_{k-1})}, z^{(k-1)(n-k)}, Q'_{(k-1)(n-k)}, v \rangle,$$

$$M'_{(k-1)(n-k)} = \langle u, P'_{(k-1)(n-k)}, y^{(k-1)(n-k)}, y, z, v \rangle.$$

(b) $(n-k)(k-1) + n + t - 2k + 3 \leq l \leq (n-k)(k-1) + n - k$. (If $k - t = 2$, then (b) is not happened.)

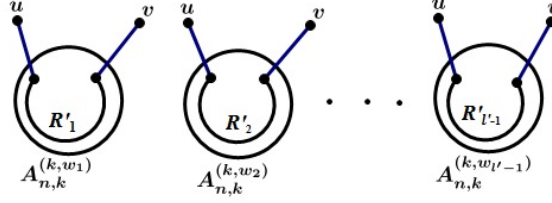


Figure 12: The paths $M_{k1}, \dots, M_{k(l'-1)}$ of step 1 of case 2.2(a) in Lemma 3.1

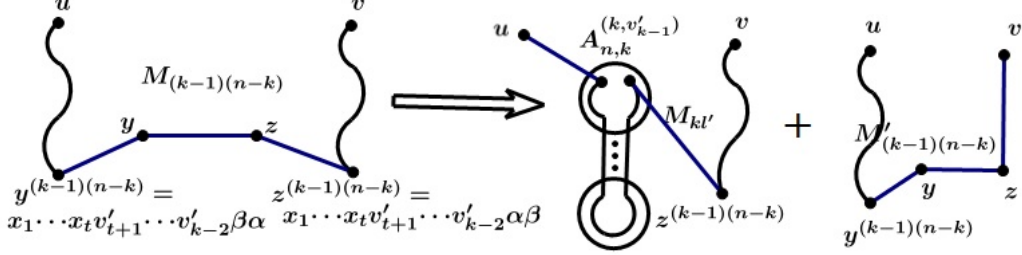


Figure 13: Illustration for step 2 of case 2.2(a) in Lemma 3.1

Let $l = (n-k)(k-1) + n + t - 2k + 2 + l'$, then $1 \leq l' \leq k - t - 2$. Thus, $U \setminus \{(U \cap V) \cup \{\alpha\}\} \neq \emptyset$. Note that $\gamma \in \langle n \rangle \setminus V$. Without loss of generality, we can assure that $\gamma = u'_{k-2}$. In order to construct the l^* -container, we need to

Step 1: By Lemma 2.4, there exists a Hamiltonian path R''_i of $A_{n,k}^{(k,w_i)}$ joining $u^{s(\alpha,w_i)}$ to $v^{s(\beta,w_i)}$ for $1 \leq i \leq n + t - 2k + 1$.

Combine figure 7 and $(n + t - 2k + 1)$ disjoint paths $M'_{k1}, M'_{k2}, \dots, M'_{k(n+t-2k+1)}$ in 14 where $M'_{ki} = \langle u, s^{s(\alpha,w_i)}, R''_i, v^{s(\beta,w_i)}, v \rangle$ for $1 \leq i \leq n + t - 2k + 1$.

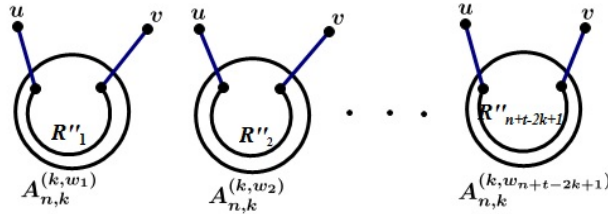


Figure 14: The paths $M'_{k1}, \dots, M'_{k(n+t-2k+1)}$ of step 1 of case 2.2(b)

Step 2: By Lemma 2.4, there exists a Hamiltonian path H_j of $A_{n,k}^{(k,u'_{t+j})}$ joining $(y^{(t+j)(n-k)})^{s(\alpha,u'_{t+j})}$ to $v^{s(\beta,u'_{t+j})}$ and a Hamiltonian path H'_j of $A_{n,k}^{(k,v'_{t+j})}$ joining $u^{s(\alpha,v'_{t+j})}$ to $(z^{(t+j)(n-k)})^{s(\beta,v'_{t+j})}$ for $1 \leq j \leq l' - 1$.

Replace $M_{(t+1)(n-k)}, \dots, M_{(t+l'-1)(n-k)}$ in part II of figure 7 by $M''_{(t+1)(n-k)}, \dots, M''_{(t+l'-1)(n-k)}$ and $M'_1, \dots, M'_{l'-1}$ as shown in figure 15 where

$$M'_j = \langle u, u^{s(\alpha,v'_{t+j})}, H'_j, (z^{(t+j)(n-k)})^{s(\beta,v'_{t+j})}, z^{(t+j)(n-k)}, Q'_{(t+j)(n-k)}, v \rangle \text{ for } 1 \leq j \leq l' - 1,$$

$$M''_{(t+j)(n-k)} = \langle u, P'_{(t+j)(n-k)}, y^{(t+j)(n-k)}, (y^{(t+j)(n-k)})^{s(\alpha,u'_{t+j})}, H_j, v^{s(\beta,u'_{t+j})}, v \rangle \text{ for } 1 \leq j \leq l' - 1.$$

Step 3: By Lemma 2.4, there exists a Hamiltonian path H_{k-1} of $A_{n,k}^{(k,u'_{k-2})}$ joining $(y^{(k-1)(n-k)})^{s(\alpha,u'_{k-2})}$ to $v^{s(\beta,u'_{k-2})}$ and a Hamiltonian path H'_{k-2} of $A_{n,k}^{(k,v'_{k-2})}$ joining $u^{s(\alpha,v'_{k-2})}$ to $(z^{(k-2)(n-k)})^{s(\beta,v'_{k-2})}$. Let $I = \{v'_{k-1}, u'_{t+l'}, u'_{t+l'+1}, \dots, u'_{k-2}\}$, then there exists a Hamiltonian path H'_{k-1} of $A_{n,k}^{(k,I)}$ joining $u^{s(\alpha,v'_{k-1})}$ to $(z^{(k-1)(n-k)})^{s(\alpha,v'_{k-1})}$.

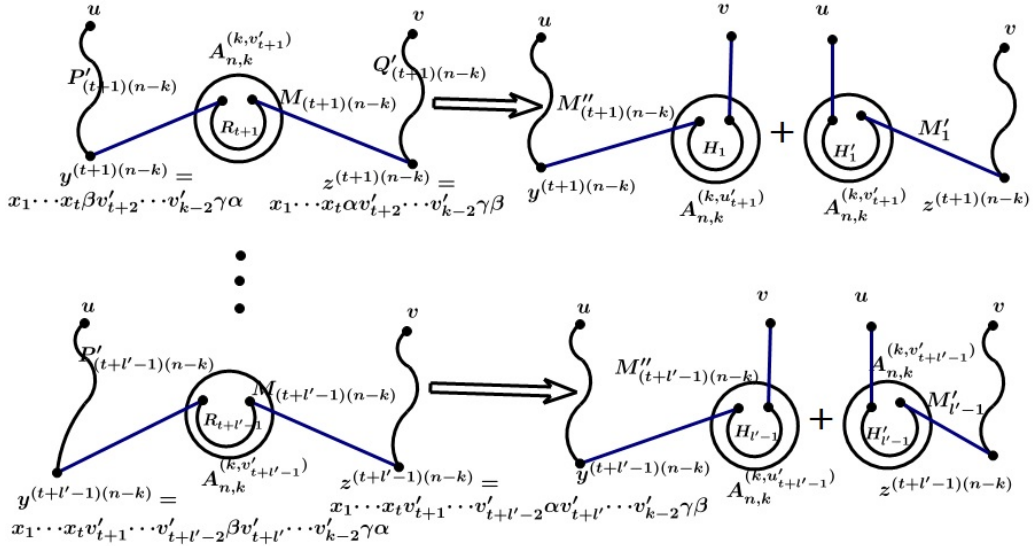


Figure 15: Illustration for step 2 of case 2.2(b) in Lemma 3.1

Replace $M_{(k-2)(n-k)}, M_{(k-1)(n-k)}$ in part II and part III of figure 7 by $M''_{(k-2)(n-k)}, M''_{(k-1)(n-k)}$ and $M'_{l'}, M'_{l'+1}$ as shown in figure 16 where

$$\begin{aligned}
 M'_{l'} &= \langle u, u^{s(\alpha, v'_{k-2})}, H'_{k-2}, (z^{(k-2)(n-k)})^{s(\beta, v'_{k-2})}, z^{(k-2)(n-k)}, Q'_{(k-2)(n-k)}, v \rangle, \\
 M'_{l'+1} &= \langle u, u^{s(\alpha, v'_{k-1})}, H'_{k-1}, (z^{(k-1)(n-k)})^{s(\beta, v'_{k-1})}, z^{(k-1)(n-k)}, Q'_{(k-1)(n-k)}, v \rangle, \\
 M''_{(k-2)(n-k)} &= \langle u, P'_{(k-2)(n-k)}, y^{(k-2)(n-k)}, y, z, v \rangle, \\
 M''_{(k-1)(n-k)} &= \langle u, P'_{(k-1)(n-k)}, y^{(k-1)(n-k)}, (y^{(k-1)(n-k)})^{s(\alpha, u'_{k-2})}, H_{k-1}, v^{s(\beta, u'_{k-2})}, v \rangle.
 \end{aligned}$$

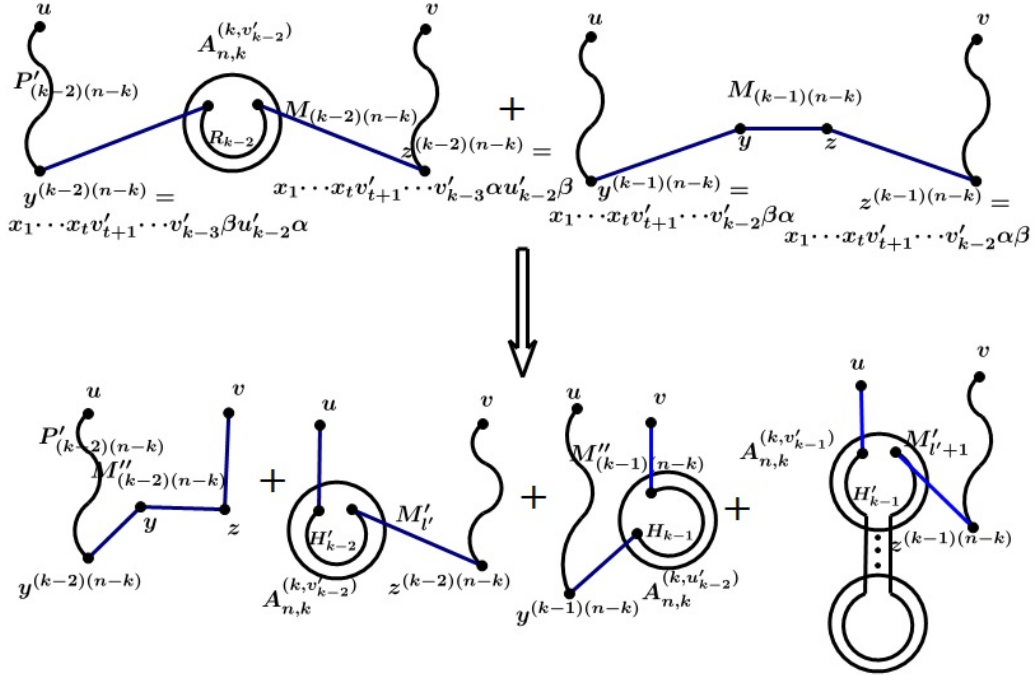


Figure 16: Illustration for step 3 of case 2.2(b) in Lemma 3.1

Case 2.3 : $U \cap V = \emptyset$.

Let $u = u_1 u_2 \cdots u_k, v = v_1 v_2 \cdots v_k, \langle n \rangle \setminus (U \cup V) = \{w_1, w_2, \dots, w_{n-2k}\}$. Set $y = v_1 v_2 \cdots v_{k-1} u_k, z = u_1 u_2 \cdots u_{k-1} v_k$. By induction, there exists an $(n-k)(k-1)^*$ -container $\{P_{ij} : 1 \leq i \leq k-1, 1 \leq j \leq n-$

$k\}$ of $A_{n,k}^{(k,u_k)}$ joining u to y and an $(n-k)(k-1)^*$ -container $\{Q_{ij} : 1 \leq i \leq n-k, 1 \leq j \leq n-k\}$ of $A_{n,k}^{(k,v_k)}$ joining z to v . We represent P_{ij} as $\langle u, P'_{ij}, y^{ij}, y \rangle, Q_{ij}$ as $\langle z, z^{ij}, Q'_{ij}, v \rangle$ for $1 \leq i \leq k-1, 1 \leq j \leq n-k$ where

$$y^{ij} = \begin{cases} y^{s(v_i, u_j)} : 1 \leq j \leq k-1, \\ y^{s(v_i, v_k)} : j = k, \\ y^{s(v_i, w_{j-k})} : k+1 \leq j \leq n-k, \end{cases} \quad z^{ij} = \begin{cases} z^{s(u_i, v_j)} : 1 \leq j \leq k-1, \\ z^{s(u_i, u_k)} : j = k, \\ z^{s(u_i, w_{j-k})} : k+1 \leq j \leq n-k. \end{cases}$$

Subcase 2.3.1: $k = 2$

Without loss of generality, let $u = 12, v = 34$.

When $n = 5$, we set the 4^* -container as :

$$\langle 12, 14, 34 \rangle, \quad \langle 12, 32, 34 \rangle, \quad \langle 12, 52, 34 \rangle, \\ \langle 12, 42, 43, 13, 53, 23, 21, 31, 51, 41, 45, 35, 15, 25, 24, 54, 34 \rangle.$$

Set the 5^* -container as:

$$\langle 12, 14, 34 \rangle, \quad \langle 12, 15, 25, 45, 35, 34 \rangle, \quad \langle 12, 32, 34 \rangle, \\ \langle 12, 52, 54, 34 \rangle, \quad \langle 12, 42, 43, 23, 13, 53, 51, 41, 31, 21, 24, 34 \rangle.$$

Set the 6^* -container as:

$$\langle 12, 14, 34 \rangle, \quad \langle 12, 13, 53, 43, 23, 24, 34 \rangle, \quad \langle 12, 15, 25, 45, 35, 34 \rangle, \\ \langle 12, 32, 34 \rangle, \quad \langle 12, 42, 41, 51, 21, 31, 34 \rangle, \quad \langle 12, 52, 54, 34 \rangle.$$

Now, let $n \geq 6$.

(a) $l = (n-k)(k-1) + 1 = n-1$.

Let $I = n \setminus \{2, 4\}$, by Lemma 2.4, there exists a Hamiltonian path R_1 of $A_{n,2}^{(2,I)}$ joining 41 to 21, we set

$$P_1 = \langle 12, 32, 34 \rangle, \quad P_2 = \langle 12, 14, 34 \rangle, \\ P_3 = \langle 12, 42, 41, R_1, 21, 24, 34 \rangle, \quad P_i = \langle 12, (i+1)2, (i+1)4, 34 \rangle \text{ for } 4 \leq i \leq n-1.$$

See figure 17 for illustration.

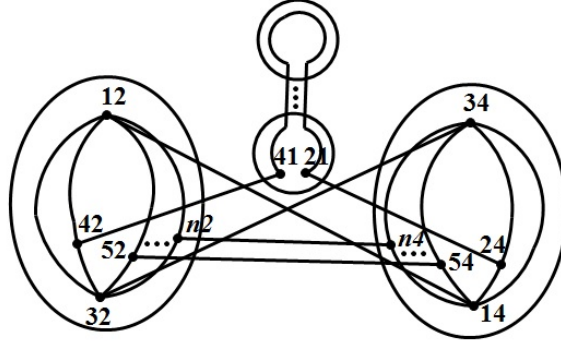


Figure 17: Illustration for subcase 2.3.1 (a) of Theorem 3.1

(b) $l = (n-k)(k-1) + 2 = n$

By Lemma 2.4, there exists a Hamiltonian path R_1 of $A_{n,2}^{(2,1)}$ joining 41 to 31. Let $I = \langle n \rangle \setminus \{1, 2, 4\}$, by Lemma 2.4, there exists a Hamiltonian path R_2 of $A_{n,k}^{(2,I)}$ joining 13 to 23. We set

$$P_1 = \langle 12, 32, 34 \rangle, \quad P_2 = \langle 12, 14, 34 \rangle, \\ P_3 = \langle 12, 42, 41, R_1, 31, 34 \rangle, \quad P_4 = \langle 12, 13, R_2, 23, 24, 34 \rangle, \\ P_i = \langle 12, i2, i4, 34 \rangle \text{ for } 5 \leq i \leq n.$$

See figure 18 for illustration.

(c) $(n-k)(k-1) + 3 = n+1 \leq l \leq (n-k)k = 2(n-2)$.

By Lemma 2.4, there exists a Hamiltonian path R_1 of $A_{n,2}^{(2,1)}$ joining 41 to 31, a Hamiltonian path R_2 of $A_{n,k}^{(2,3)}$ joining 13 to 23, and a Hamiltonian path R_j of $A_{n,k}^{(2,j)}$ joining $1j$ to $3j$ for $5 \leq j \leq l-n+3$. Let $I = \langle n \rangle \setminus \{1, 2, \dots, l-n+3\}$. By Lemma 2.4, there exists a Hamiltonian path H_{l-n+4} of $A_{n,k}^{(2,l-n+4)}$ joining $1(l-n+4)$ to $3(l-n+4)$. We set

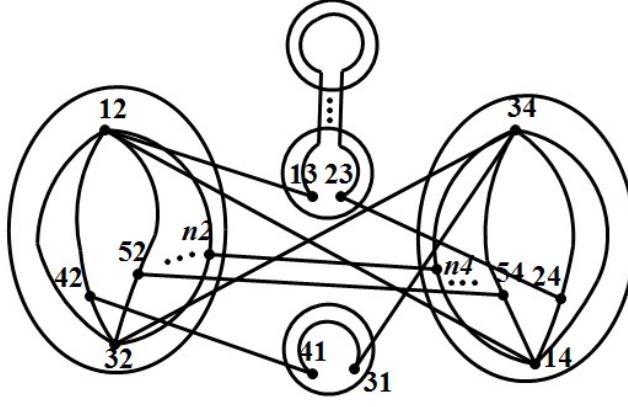


Figure 18: Illustration for subcase 2.3.1 (b) of Theorem 3.1

$$\begin{aligned}
P_1 &= \langle 12, 32, 34 \rangle, \\
P_2 &= \langle 12, 14, 34 \rangle, \\
P_3 &= \langle 12, 42, 41, R_1, 31, 34 \rangle, \\
P_4 &= \langle 12, 13, R_2, 23, 24, 34 \rangle, \\
P_i &= \langle 12, i2, i4, 34 \rangle \text{ for } 5 \leq i \leq n, \\
P_j &= \langle 12, 1j, R_j, 3j, 34 \rangle \text{ for } 5 \leq j \leq l - n + 3, \\
P_{l-n+4} &= \langle 12, 1(l - n + 4), H_{l-n+4}, 3(l - n + 4), 34 \rangle.
\end{aligned}$$

See figure 19 for illustration.

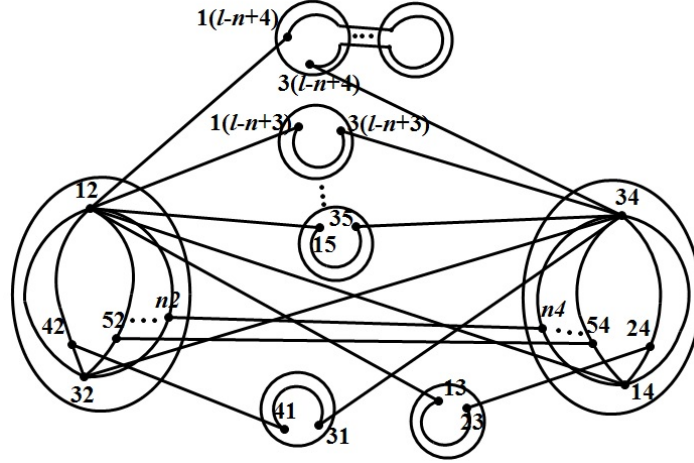


Figure 19: Illustration for subcase 2.3.1 (c) of Theorem 3.1

Subcase 2.3.2: $k \geq 3$.

(a) $l = (n - k)(k - 1) + 1$.

Let

$$\begin{aligned}
A_1 &= \{(y^{12})^{s(u_k, v_1)}, \dots, (y^{1(k-1)})^{s(u_k, v_1)}, (y^{1(k+1)})^{s(u_k, v_1)}, \dots, (y^{1(n-k)})^{s(u_k, v_1)}\}, \\
B_1 &= \{(z^{12})^{s(v_k, v_1)}, \dots, (z^{1(k-1)})^{s(v_k, v_1)}, (z^{1(k+1)})^{s(v_k, v_1)}, \dots, (z^{1(n-k)})^{s(v_k, v_1)}\}, \\
A_i &= \{(y^{i1})^{s(u_k, v_i)}, \dots, (y^{i(n-k)})^{s(u_k, v_i)}\} \text{ for } 2 \leq i \leq k - 1, \\
B_i &= \{(z^{i1})^{s(v_k, v_i)}, \dots, (z^{i(i-1)})^{s(v_k, v_i)}, (z^{i(i-1)+1})^{s(v_k, v_i)}, \dots, (z^{i(n-k)})^{s(v_k, v_i)}\} \\
&\quad \text{for } 2 \leq i \leq k - 1.
\end{aligned}$$

We partite $A_{n,k}^{(k, v_i)}$ to $\cup_{j \in \langle n \rangle \setminus \{v_i\}} A_{n,k}^{(i, j)(k, v_i)}$. By Lemma 2.5, there exists $(n - k - 2)$ disjoint paths $H_{12}, H_{13}, \dots, H_{1(k-1)}, H_{1(k+1)}, \dots, H_{1(n-k)}$ of $A_{n,k}^{(k, v_1)}$ from A_1 to B_1 such that $V(\cup_{j \in \langle n-k \rangle \setminus \{1, k\}} H_{1j}) = V(A_{n,k}^{(k, v_1)})$ and $H_{1j} = \langle (y^{1j})^{s(u_k, v_1)}, H_{1j}, (z^{1j})^{s(v_k, v_1)} \rangle$ for $j \in \langle n - k \rangle \setminus \{1, k\}$. For $2 \leq i \leq k - 1$, there

exists $(n - k)$ disjoint paths $H_{i1}, H_{i2}, \dots, H_{i(n-k)}$ of $A_{n,k}^{(k,v_i)}$ from A to B such that $V(\cup_{j=1}^{n-k} H_{ij}) = V(A_{n,k}^{(k,v_i)})$ and $H_{ii} = \langle (y^{ii})^{s(u_k, v_i)}, H_{ii}, (z^{(i-1)(i-1)})^{s(v_k, v_i)} \rangle$, $H_{ij} = \langle (y^{ij})^{s(u_k, v_i)}, H_{ij}, (z^{ij})^{s(v_k, v_i)} \rangle$ for $j \in \langle n - k \rangle \setminus \{i\}$. Let $I = \langle n \rangle \setminus \{v_1, \dots, v_{k-1}, v_k, u_k\}$, by Lemma 2.4, there exists a Hamiltonian path H_{1k} of $A_{n,k}^{(k,I)}$ joining $(y^{1k})^{s(u_k, u_1)}$ to $(z^{1k})^{s(v_k, u_1)}$. We set

$$\begin{aligned} M_{11} &= \langle u, P'_{11}, y^{11}, y, v \rangle, \\ M_{1k} &= \langle u, P'_{1k}, y^{1k}, (y^{1k})^{s(u_k, u_1)}, H_{1k}, (z^{1k})^{s(v_k, u_1)}, z^{1k}, Q'_{1k}, v \rangle, \\ M_{1t} &= \langle u, P'_{1t}, y^{1t}, (y^{1t})^{s(u_k, v_1)}, H_{1t}, (z^{1t})^{s(v_k, v_1)}, z^{1t}, Q'_{1t}, v \rangle \text{ for } 2 \leq t \leq n - k \text{ and } t \neq k, \\ M_{ii} &= \langle u, P'_{ii}, y^{ii}, (y^{ii})^{s(u_k, v_i)}, H_{ii}, (z^{(i-1)(i-1)})^{s(v_k, v_i)}, z^{(i-1)(i-1)}, Q'_{(i-1)(i-1)}, v \rangle \text{ for } 2 \leq i \leq k - 1, \\ M_{ij} &= \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(u_k, v_i)}, H_{ij}, (z^{ij})^{s(v_k, v_i)}, z^{ij}, Q'_{ij}, v \rangle \text{ for } 2 \leq i \leq k - 1, \\ &\quad 1 \leq j \leq n - k \text{ and } j \neq i, \\ M_{k1} &= \langle u, z, z^{(k-1)(k-1)}, Q'_{(k-1)(k-1)}, v \rangle. \end{aligned}$$

See figure 20 for illustration.

(b) $(n - k)(k - 1) + 2 \leq l \leq (n - k)(k - 1) + n - 2k + 1$. (If $n - 2k = 0$, then (b) does not occur)
Let $l = (n - k)(k - 1) + 1 + l'$ and let

$$\begin{aligned} A_1 &= \{(y^{12})^{s(u_k, v_1)}, \dots, (y^{1(n-k)})^{s(u_k, v_1)}\}, \\ B_1 &= \{(z^{12})^{s(v_k, v_1)}, \dots, (z^{1(n-k)})^{s(v_k, v_1)}\}. \end{aligned}$$

We partite $A_{n,k}^{(k,v_1)}$ to $\cup_{j \in \langle n \rangle \setminus \{v_1\}} A_{n,k}^{(1,j)(k,v_1)}$. To construct the l^* -container, we need

Step 1: By Lemma 2.5, there exists $(n - k - 1)$ disjoint paths $H'_{12}, H'_{13}, \dots, H'_{1(n-k)}$ of $A_{n,k}^{(k,v_1)}$ from A_1 to B_1 such that $V(\cup_{j=2}^{n-k} H'_{1j}) = V(A_{n,k}^{(k,v_1)})$ and $H'_{1j} = \langle (y^{1j})^{s(u_k, v_1)}, H'_{1j}, (z^{1j})^{s(v_k, v_1)} \rangle$ for $2 \leq j \leq n - k$.

Replace paths $M_{12}, M_{13}, \dots, M_{1(n-k)}$ in figure 20 by $M'_{12}, M'_{13}, \dots, M'_{1(n-k)}$ as shown in figure 21 where

$$M'_{1j} = \langle u, P'_{1j}, y^{1j}, (y^{1j})^{s(u_k, v_1)}, H'_{1j}, (z^{1j})^{s(v_k, v_1)}, z^{1j}, Q'_{1j}, v \rangle \text{ for } j \in \langle n - k \rangle \setminus \{1\}.$$

Step 2: Let $I = \langle n \rangle \setminus \{v_1, \dots, v_{k-1}, v_k, u_k, w_1, \dots, w_{l'-1}\}$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k,w_i)}$ joining $u^{s(u_k, w_i)}$ to $v^{s(v_k, w_i)}$ for $1 \leq i \leq l' - 1$ and a Hamiltonian path $R_{l'}$ of $A_{n,k}^{(k,I)}$ joining $(u)^{s(u_k, w_{l'})}$ to $v^{s(v_k, w_{l'})}$. Combine figure 20 and l' paths $M_1, M_2, \dots, M_{l'}$ in figure 22 where

$$M_i = \langle u, u^{s(u_k, w_i)}, R_i, v^{s(v_k, w_i)}, v \rangle \text{ for } 1 \leq i \leq l'.$$

(c) $(n - k)(k - 1) + n - 2k + 2 \leq (n - k)(k - 1) + n - k$.

Let $l = (n - k)(k - 1) + n - 2k + 1 + l'$, then $1 \leq l' \leq k - 1$.

To construct the l^* -container, we need:

Step 1: By Lemma 2.4, there exists a Hamiltonian path R'_i of $A_{n,k}^{(k,w_i)}$ joining $u^{s(u_k, w_i)}$ to $v^{s(v_k, w_i)}$ for $1 \leq i \leq n - 2k$.

Combine figure 20 and $(n - 2k)$ disjoint paths $M'_1, M'_2, \dots, M'_{n-2k}$ as shown in figure 23 where $M'_i = \langle u, u^{s(u_k, w_i)}, R'_i, v^{s(v_k, w_i)}, v \rangle$ for $1 \leq i \leq n - 2k$.

Step 2: Notice $k \geq 3, n \geq 2k \geq 6$, then $\frac{(n-3)!}{(n-k-1)!} \geq 3$, so there exists at least 3 edges between $A_{n,k}^{(1,u_1)(k,v_1)}$ and $A_{n,k}^{(1,u_1)(k,u_{k-1})}$. Let (x^1, x^2) be an edge of $E(A_{n,k}^{(1,u_1)(k,v_1)}, A_{n,k}^{(1,u_1)(k,u_{k-1})})$ such that $x^1 \in A_{n,k}^{(1,u_1)(k,v_1)}, x^2 \in A_{n,k}^{(1,u_1)(k,u_{k-1})}$ and $x^1 \neq u^{s(u_k, v_1)}, x^2 \neq v^{s(v_k, u_{k-1})}$. By Lemma 2.4, there exists a Hamiltonian path R of $A_{n,k}^{(k,u_{k-1})}$ joining x^2 to $v^{s(v_k, u_{k-1})}$. Let

$$\begin{aligned} A_1 &= \{u^{s(u_k, v_1)}, (y^{12})^{s(u_k, v_1)}, \dots, (y^{1(n-k)})^{s(u_k, v_1)}\}, \\ B_1 &= \{x^1, (z^{12})^{s(v_k, v_1)}, \dots, (z^{1(n-k)})^{s(v_k, v_1)}\}. \end{aligned}$$

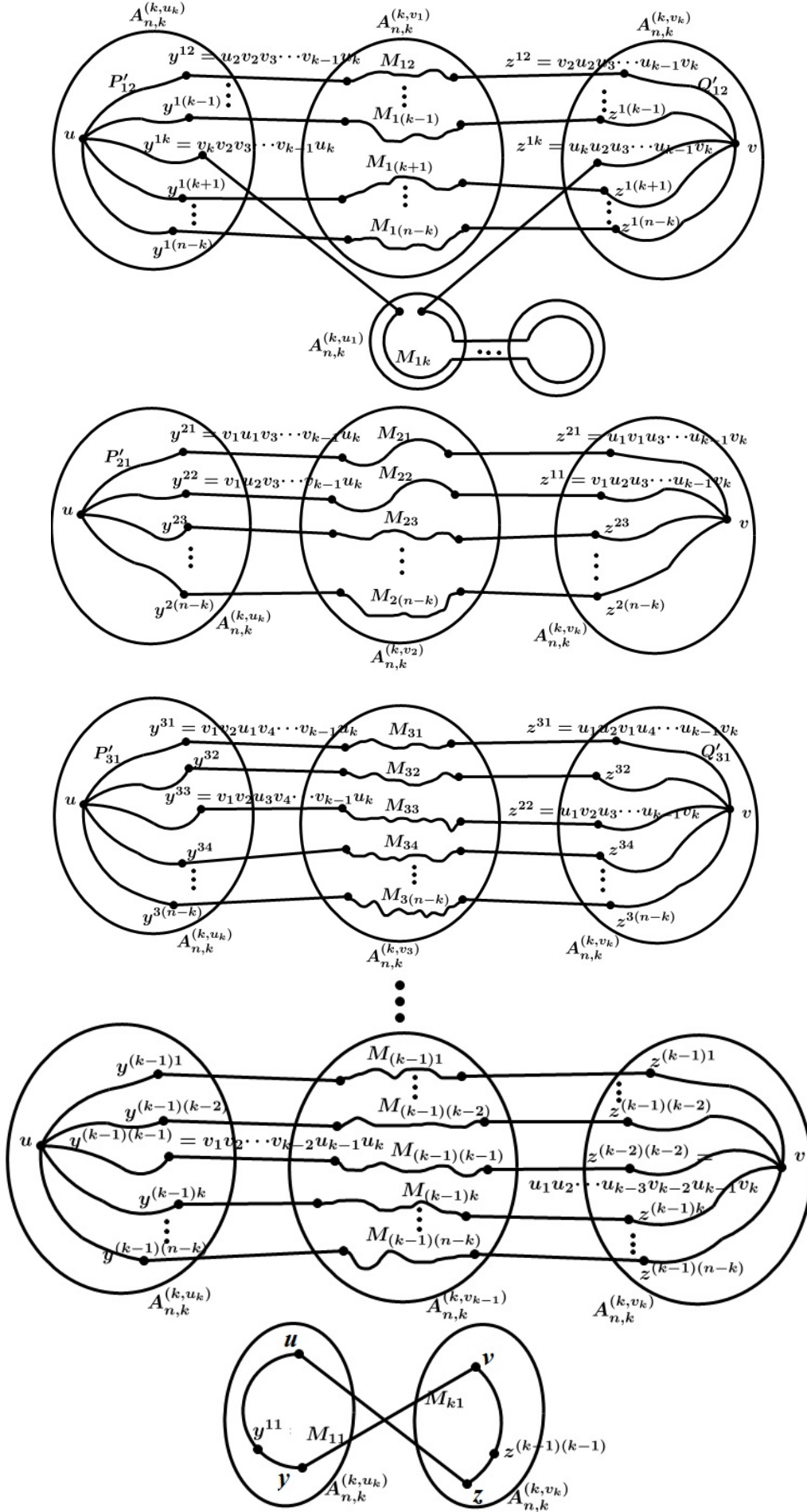


Figure 20: Illustration for subcase 2.3.2 (a) of Theorem 3.1

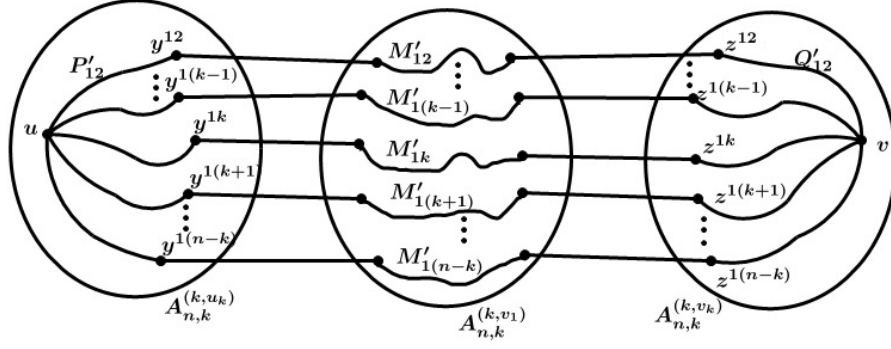


Figure 21: Illustration for step 1 of case 2.3(b) in Lemma 3.1

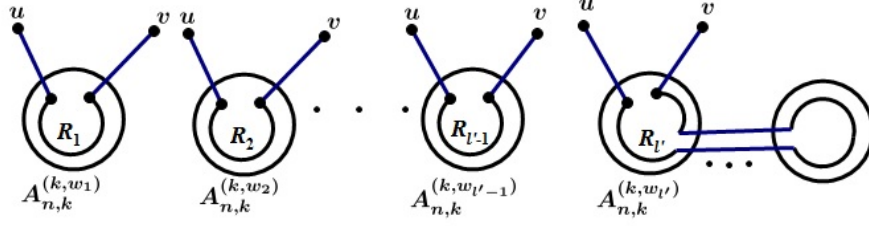


Figure 22: The paths $M_1, \dots, M_{l'}$ of step 2 of case 2.3(b) in Lemma 3.1

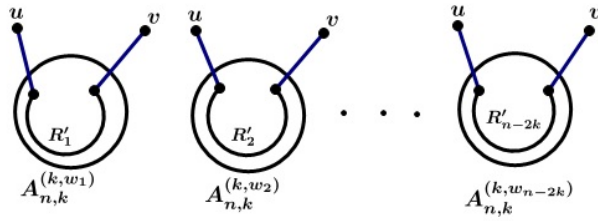


Figure 23: The paths M'_1, \dots, M'_{n-2k} of step 1 of case 2.3(c) in Lemma 3.1

By Lemma 2.5, there exists $(n - k)$ disjoint paths $H''_{11}, H''_{12}, \dots, H''_{1(n-k)}$ of $A_{n,k}^{(k,v_1)}$ from A_1 to B_1 such that $V(\cup_{i=1}^{n-k} H''_{1i}) = V(A_{n,k}^{(k,v_1)})$ and $H''_{11} = \langle u^{s(u_k,v_1)}, H''_{11}, x^1 \rangle$, $H''_{1i} = \langle (y^{1i})^{s(u_k,v_1)}, H''_{1i}, (z^{1i})^{s(v_k,v_1)} \rangle$ for $2 \leq i \leq n - k$.

Replace paths $M_{12}, M_{13}, M_{1(n-k)}$ in figure 20 by paths $M''_{11}, M''_{12}, M''_{13}, \dots, M''_{1(n-k)}$ as shown in figure 24 where

$$\begin{aligned} M''_{11} &= \langle u, u^{s(u_k,v_1)}, H''_{11}, x^1, x^2, R, v^{s(v_k,u_{k-1})}, v \rangle, \\ M''_{1i} &= \langle u, P'_{1i}, y^{1i}, (y^{1i})^{s(u_k,v_1)}, H''_{1i}, (z^{1i})^{s(v_k,v_1)}, z^{1i}, Q'_{1i}, v \rangle \text{ for } 2 \leq i \leq n - k. \end{aligned}$$

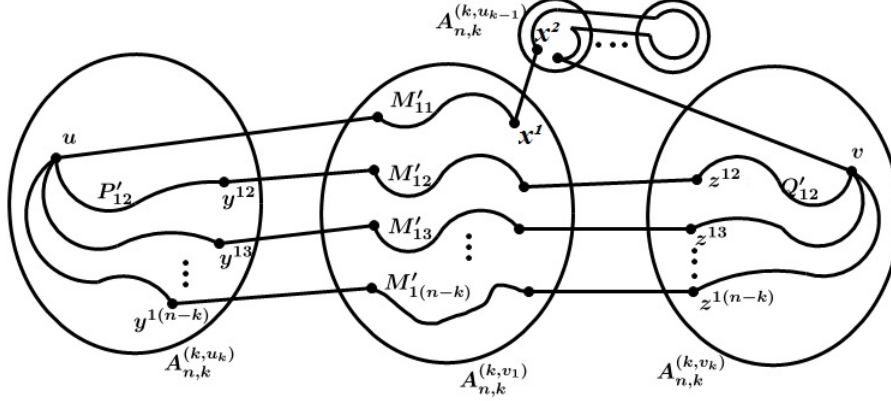


Figure 24: Illustration for step 2 of case 2.3(c) in Lemma 3.1

Step 3: Let

$$\begin{aligned} A_i &= \{(y^{i1})^{s(u_k,v_i)}, \dots, (y^{i(i-1)})^{s(u_k,v_i)}, u^{s(u_k,v_i)}, (y^{i(i+1)})^{s(u_k,v_i)}, \dots, (y^{i(n-k)})^{s(u_k,v_i)}\} \text{ for } 2 \leq i \leq l' \\ B^i &= \{(z^{i1})^{s(v_k,v_i)}, \dots, (z^{i(i-1)})^{s(v_k,v_i)}, (z^{i(i-1)(i-1)})^{s(v_k,v_i)}, (z^{i(i+1)})^{s(v_k,v_i)}, \dots, (z^{i(n-k)})^{s(v_k,v_i)}\} \\ &\quad \text{for } 2 \leq i \leq l'. \end{aligned}$$

For $2 \leq i \leq l'$, we partite $A_{n,k}^{(k,v_i)}$ to $\cup_{r \in \langle n \rangle \setminus \{v_i\}} A_{n,k}^{(i,r)}$. By Lemma 2.5, there exists $(n - k)$ disjoint paths $H''_{i1}, H''_{i2}, \dots, H''_{i(n-k)}$ of $A_{n,k}^{(k,v_i)}$ from A^i to B^i such that $V(\cup_{j=1}^{n-k} H''_{ij}) = V(A_{n,k}^{(k,v_i)})$ and $H''_{ii} = \langle u^{s(u_k,v_i)}, H''_{ii}, (z^{(i-1)(i-1)})^{s(v_k,v_i)} \rangle$, $H''_{ij} = \langle (y^{ij})^{s(u_k,v_i)}, H''_{ij}, (z^{ij})^{s(v_k,v_i)} \rangle$ for $j \in \langle n - k \rangle \setminus \{i\}$. By Lemma 2.4, there exists a Hamiltonian path H_{ki} of $A_{n,k}^{(k,u_{i-1})}$ joining $(y^{ii})^{s(u_k,u_{i-1})}$ to $v^{s(v_k,u_{i-1})}$ for $2 \leq i \leq l'$.

Replace paths $M_{i1}, M_{i2}, \dots, M_{i(n-k)}$ in figure 20 by $M_{ki}, M''_{i1}, M''_{i2}, \dots, M''_{i(n-k)}$ for $2 \leq i \leq l'$ as shown in figure 25 where

$$\begin{aligned} M_{ki} &= \langle u, P'_{ii}, y^{ii}, (y^{ii})^{s(u_k,u_{i-1})}, H_{ki}, v^{s(v_k,u_{i-1})}, v \rangle, \\ M''_{ii} &= \langle u, u^{s(u_k,v_i)}, H''_{ii}, (z^{(i-1)(i-1)})^{s(v_k,v_i)}, z^{(i-1)(i-1)}, Q'_{(i-1)(i-1)}, v \rangle, \\ M''_{ij} &= \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(u_k,v_i)}, H''_{ij}, (z^{ij})^{s(v_k,v_i)}, z^{ij}, Q'_{ij}, v \rangle. \end{aligned}$$

for $2 \leq i \leq l', 1 \leq j \leq n - k$ and $j \neq i$.

□

Theorem 3.2 $A_{n,k}$ is super spanning connected for $n \geq 4, n - k \geq 2$.

Proof: We prove the theorem by induction.

Basis step: It is known that $A_{n,1}$ is isomorphic to the complete graph K_n and by Lemma 2.3, $A_{4,2}$ is super spanning connected. Then, the result holds for $A_{n,1}$ and $A_{4,2}$.

Induction step: Suppose $A_{n-1,k-1}$ is super spanning connected. We need to prove that $A_{n,k}$ is super spanning connected for $n \geq 5, n - k \geq 2$. By Lemma 2.1, $A_{n,k}$ is 1*-connected and 2*-connected. By Lemma 3.1, $A_{n,k}$ is l^* -connected for $(n - k)(k - 1) + 1 \leq l \leq (n - k)k$. Now, we need to construct a l^* -container of $A_{n,k}$ joining any two distinct vertices u and v for $3 \leq l \leq (n - k)(k - 1)$. We use U to denote the set $\{(u)_i \mid 1 \leq i \leq k\}$ and use V to denote the set $\{(v)_i \mid 1 \leq i \leq k\}$.

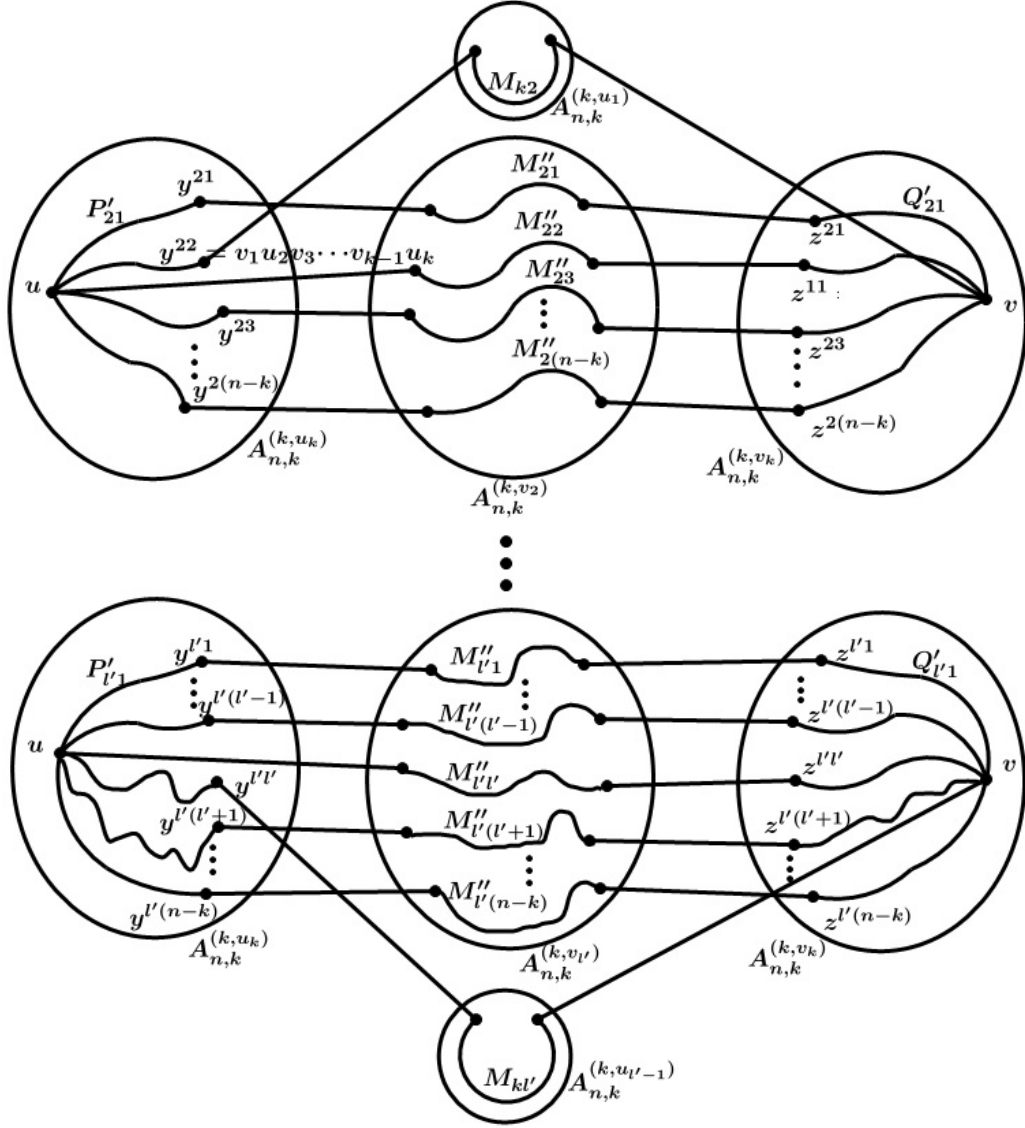


Figure 25: Illustration for step 3 of case 2.3(c) in Lemma 3.1

Case 1: $\{i \mid (u)_i = (v)_i : 1 \leq i \leq k\} \neq \emptyset$.

Without loss of generality, let $(u)_k = (v)_k = \alpha$. By induction, there is an l^* -container $\{P_1, P_2, \dots, P_l\}$ of $A_{n,k}^{(k,\alpha)}$ joining u to v . Hence, we can represent P_l as $\langle u, y, P'_l, v \rangle$. Note that $|\{(u)_i : 1 \leq i \leq k\} \cup \{(y)_i : 1 \leq i \leq k\}| = k + 1$ and $n - k \geq 2$. Suppose $\beta \in \langle n \rangle \setminus \{\{(u)_i : 1 \leq i \leq k\} \cup \{(y)_i : 1 \leq i \leq k\}\}$. By Lemma 2.4, there exists a Hamiltonian path H of $A_{n,k}^{(k,\langle n \rangle \setminus \{\alpha\})}$ joining $u^{s(\alpha,\beta)}$ to $y^{s(\alpha,\beta)}$. We set $P'_l = \langle u, u^{s(\alpha,\beta)}, H, y^{s(\alpha,\beta)}, y, P'_l, v \rangle$. Obviously, $\{P_1, P_2, \dots, P_{l-1}, P'_l\}$ is a l^* -container of $A_{n,k}$ joining u to v . See figure 26 for illustration.

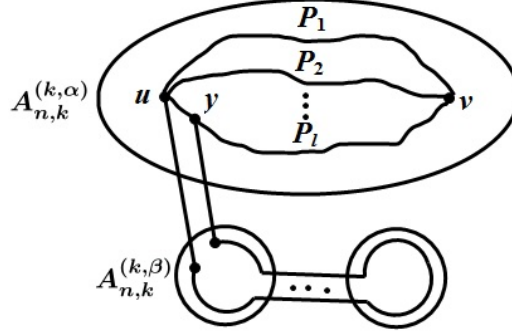


Figure 26: Illustration for case 1 of Theorem 3.2

Case 2: $\{i \mid (u)_i = (v)_i : 1 \leq i \leq k\} = \emptyset$.

Case 2.1: $U \neq V$.

Without loss of generality, we can assume that $(u)_k = \alpha \notin V$. We partite $A_{n,k}$ to $\cup_{i \in \langle n \rangle} A_{n,k}^{(k,i)}$. Suppose $u = u_1 u_2 \dots u_{k-1} \alpha, v = v_1 v_2 \dots v_{k-1} \beta$. Set $y = v_1 v_2 \dots v_{k-1} \alpha$, then $y \neq u$ and $(y, v) \in E(A_{n,k})$. By induction, there exists an l^* -container $\{P_1, P_2, \dots, P_l\}$ of $A_{n,k}^{(k,\alpha)}$ joining u to y . We represent P_i as $\langle u, P'_i, y^i, y \rangle$ for $1 \leq i \leq l$. Without loss of generality, we can assume that $V(P_l) \leq V(P_i)$ for $1 \leq i \leq l$. Suppose $|\{y^i \mid \beta \in \{(y^i)_j : 1 \leq j \leq k-1\}, 1 \leq i \leq l-1\}| = m$, then $0 \leq m \leq \min\{l-1, k-1\}$.

Subcase 2.1.1 $m = 0$.

For all $1 \leq i \leq l-1$, we have $\beta \notin \{(y^i)_j : 1 \leq j \leq k-1\}$. Let $z^i = (y^i)^{s(\alpha,\beta)}$ for $1 \leq i \leq l-2$. Then, $z^i \in A_{n,k}^{(k,\beta)}$ and $(z^i, v) \in E(A_{n,k})$ for $1 \leq i \leq l-2$. Note that $l-3 \leq (n-k)(k-1)-3$, by Lemma 2.1, there exists a Hamiltonian path R of $A_{n,k}^{(k,\beta)} \setminus \{z^1, z^2, \dots, z^{l-3}\}$ joining z^{l-2} to v . Since y^{l-1} and y differ in exactly one position, we can assume that $y^{l-1} = v_1 \dots v_{r-1} x v_{r+1} \dots v_{k-1} \alpha$ where $x \in \langle n \rangle \setminus \{v_1, v_2, \dots, v_{k-1}, \alpha, \beta\}$. Let $I = \langle n \rangle \setminus \{\alpha, \beta\}$. By Lemma 2.4, there exists a Hamiltonian path H of $A_{n,k}^{(k,I)}$ joining $(y^{l-1})^{s(\alpha,v_r)}$ to $v^{s(\beta,x)}$. We set

$$\begin{aligned} M_i &= \langle u, P'_i, y^i, z^i, v \rangle \text{ for } 1 \leq i \leq l-3, \\ M_{l-2} &= \langle u, P'_{l-2}, y^{l-2}, z^{l-2}, R, v \rangle, \\ M_{l-1} &= \langle u, P'_{l-1}, y^{l-1}, (y^{l-1})^{s(\alpha,v_r)}, H, v^{s(\beta,x)}, v \rangle, \\ M_l &= \langle u, P'_l, y^l, y, v \rangle. \end{aligned}$$

Obviously, $\{M_1, M_2, \dots, M_l\}$ forms an l^* -container of $A_{n,k}$. See figure 27 for illustration.

Subcase 2.1.2 $0 < m < l-1$.

Without loss of generality, we can assume that $y^i = v_1 \dots v_{i-1} \beta v_{i+1} \dots v_{k-1} \alpha$ for $1 \leq i \leq m$. Set $z^i = v_1 \dots v_{i-1} \alpha v_{i+1} \dots v_{k-1} \beta$ for $1 \leq i \leq m$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k,v_i)}$ joining $(y^i)^{s(\alpha,v_i)}$ to $(z^i)^{s(\beta,v_i)}$ for $1 \leq i \leq m-1$. Since $n-k \geq 2$, suppose $\gamma \in \langle n \rangle \setminus \{v_1, \dots, v_{k-1}, \alpha, \beta\}$. Let $I = \langle n \rangle \setminus \{v_1, \dots, v_{m-1}, \alpha, \beta\}$. By Lemma 2.4, there exists a Hamiltonian path R_m of $A_{n,k}^{(k,I)}$ joining $(y^m)^{s(\alpha,v_m)}$ to $v^{s(\beta,\gamma)}$. Notice that $\beta \notin \{(y^i)_j : 1 \leq j \leq k-1\}$ for $m+1 \leq i \leq l-1$. We set $z^i = (y^i)^{s(\alpha,\beta)}$ for $m+1 \leq i \leq l-1$. Then, $(y^i, z^i) \in E(A_{n,k})$ for $m+1 \leq i \leq l-1$. Since $l-3 \leq (n-k)(k-1)-3$, by Lemma 2.1, there exists a Hamiltonian path R of $A_{n,k}^{(k,\beta)} \setminus \{v_1, \dots, v_{m-1}, v_{m+1}, \dots, v_{l-2}\}$ joining z^{l-1} to v . We set

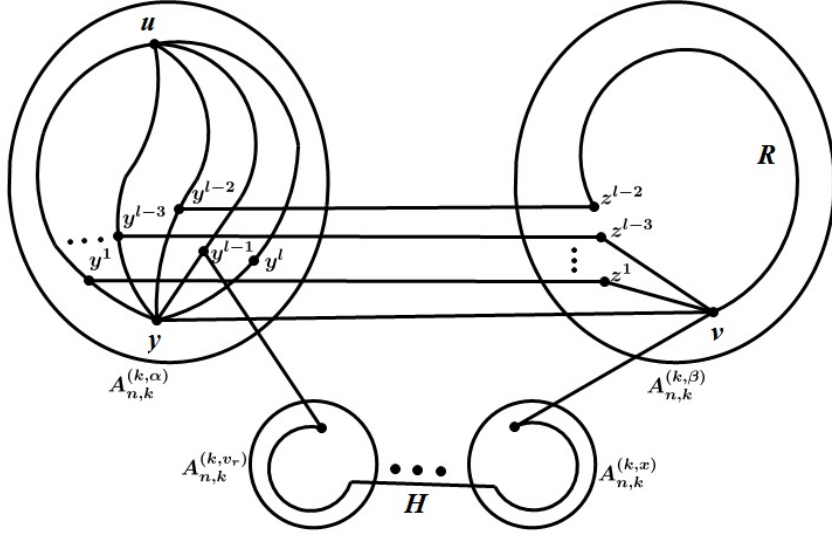


Figure 27: Illustration for subcase 2.1.1 of Theorem 3.2

$$M_i = \langle u, P'_i, y^i, (y^i)^{s(\alpha, v_i)}, R_i, (z^i)^{s(\beta, v_i)}, z^i, v \rangle \text{ for } 1 \leq i \leq m-1,$$

$$M_m = \langle u, P'_m, y^m, (y^m)^{s(\alpha, v_m)}, R_m, v^{s(\beta, \gamma)}, v \rangle,$$

$$M_{m+j} = \langle u, P'_{m+j}, y^{m+j}, z^{m+j}, z \rangle \text{ for } 1 \leq j \leq l-m-2,$$

$$M_{l-1} = \langle u, P'_{l-1}, y^{l-1}, z^{l-1}, R, v \rangle,$$

$$M_l = \langle u, P'_l, y^l, y, v \rangle.$$

Obviously, $\{M_1, M_2, \dots, M_l\}$ forms an l^* -container of $A_{n,k}$. See figure 28 for illustration.

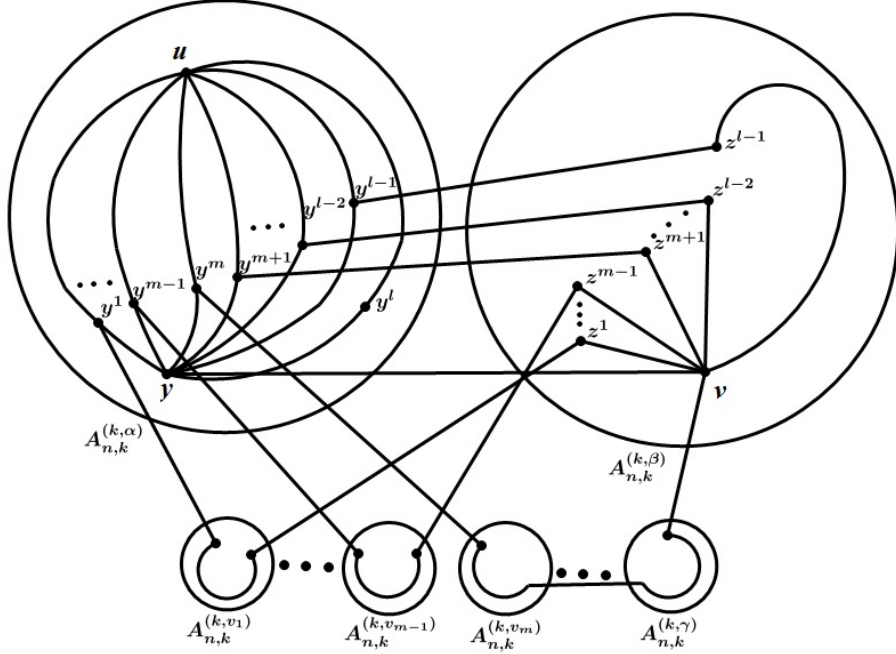


Figure 28: Illustration for subcase 2.1.2 of Theorem 3.2

Subcase 2.1.3 $m = l-1$.

Without loss of generality, we can assume $y^i = v_1 \dots v_{i-1} \alpha v_{i+1} \dots v_{k-1} \beta$ for $1 \leq i \leq l-1$. Set $z^i = v_1 \dots v_{i-1} \alpha v_{i+1} \dots v_{k-1} \beta$ for $1 \leq i \leq l-1$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k, v_i)}$ joining $(y^i)^{s(\alpha, v_i)}$ to $(z^i)^{s(\beta, v_i)}$ for $1 \leq i \leq l-2$. Since $n-k \geq 2$, suppose $\gamma \in \langle n \rangle \setminus \{v_1, \dots, v_{k-1}, \alpha, \beta\}$. Let $I = \langle n \rangle \setminus \{v_1, \dots, v_{l-2}, \alpha, \beta\}$. By Lemma 2.4, there exists a Hamiltonian path R_m of $A_{n,k}^{(k, I)}$ joining $(y^{l-1})^{s(\alpha, v_{l-1})}$ to $(z^{l-1})^{s(\beta, \gamma)}$. Since $1 \leq l-2 = m-1 \leq k-2 \leq (n-k)(k-1) - 3$, by Lemma 2.4,

there exists a Hamiltonian path R of $A_{n,k}^{(k,\beta)}$ joining z^{l-1} to v . We set

$$M_i = \langle u, P'_i, y^i, (y^i)^{s(\alpha, v_i)}, R_i, (z^i)^{s(\beta, v_i)}, z^i, v \rangle \text{ for } 1 \leq i \leq l-2,$$

$$M_{l-1} = \langle u, P'_{l-1}, y^{l-1}, (y^{l-1})^{s(\alpha, v_{l-1})}, R_{l-1}, (z^{l-1})^{s(\beta, v_{l-1})}, z^{l-1}, R, v \rangle,$$

$$M_l = \langle u, P'_l, y^l, y, v \rangle.$$

Obviously, $\{M_1, M_2, \dots, M_l\}$ forms an l^* -container of $A_{n,k}$. See figure 29 for illustration.

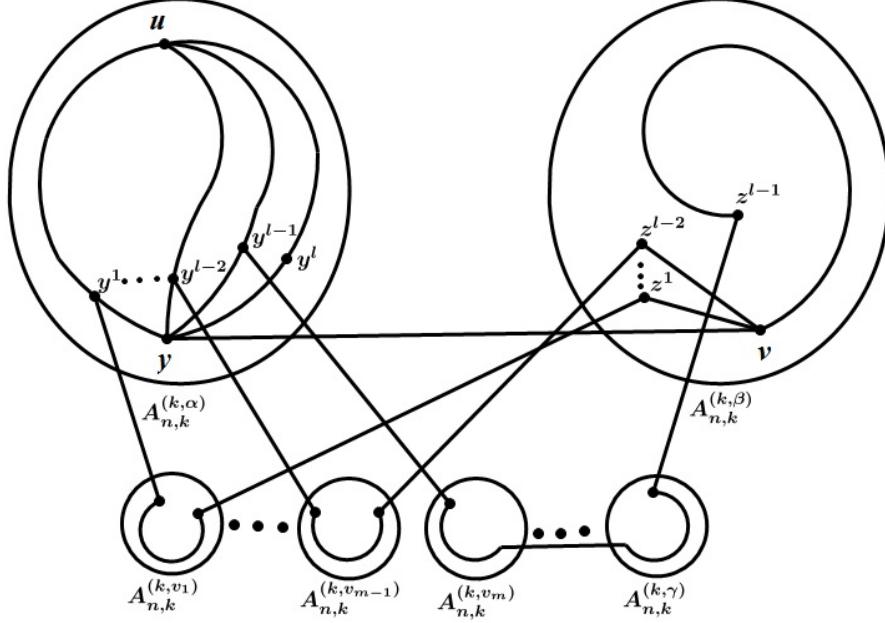


Figure 29: Illustration for subcase 2.1.3 of Theorem 3.2

Case 2.2 $U = V$.

Subcase 2.2.1 $k = 2$.

Without loss of generality, we can assume that $u = 12, v = 21, n \geq 5$. By Lemma 2.4, there exists a Hamiltonian path R_1 of $A_{n,2}^{(2,2)}$ joining 12 to $n2$, a Hamiltonian path R_2 of $A_{n,2}^{(2,1)}$ joining $(n-1)1$ to 21 and a Hamiltonian path R_3 of $A_{n,k}^{(2,3)}$ joining $n3$ to $(n-1)3$. Additionally, there exists a Hamiltonian path R_i of $A_{n,2}^{(2,i)}$ joining $1i$ to $2i$ for $4 \leq i \leq l-1$ and a Hamiltonian path R of $A_{n,2}^{(k, \{l, l+1, \dots, n\})}$ joining $1l$ to $2n$. We set

$$M_1 = \langle 12, R_1, n3, R_3, (n-1)3, (n-1)1, R_2, 21 \rangle,$$

$$M_i = \langle 12, 1(i+2), R_{i+2}, 2(i+2), 21 \rangle \text{ for } 2 \leq i \leq l-1,$$

$$M_l = \langle 12, 1l, R, 2n, 21 \rangle.$$

Obviously, $\{M_1, M_2, \dots, M_l\}$ forms an l^* -container of $A_{n,2}$. See figure 30 for illustration.

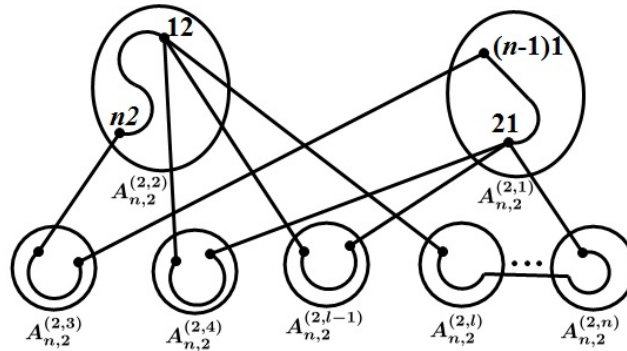


Figure 30: Illustration for subcase 2.2.1 of Theorem 3.2

Subcase 2.2.2 $k \geq 3$.

Without loss of generality, we can assume that $v = 12 \cdots k$, $u = u_1 u_2 \cdots u_{k-1} r$ and $r = 1$. Let $y = k23 \cdots (k-1)1$. By induction, there exists an l^* -container $\{P_1, P_2, \dots, P_l\}$ joining u to y . We represent P_i as $\langle u, P'_i, y^i, y \rangle$ for $1 \leq i \leq l$.

If $|\{y^i \mid (y^i)_1 \neq k\}| = 1$, we may assume that $(y^l)_1 \neq k$. If $|\{y^i \mid (y^i)_1 \neq k\}| = 2$, we may assume that $(y^{l-1})_1 \neq k, (y^l)_1 \neq k$.

Let $Y = Y^1 \cup Y^2 \cup \dots \cup Y^{k-1}$ be the neighbors of y in P_i for $1 \leq i \leq l-2$ where $Y^j = \{y^{j1}, y^{j2}, \dots, y^{jn_j}\}$ are obtained by switch the j th coordinate of y . Then, $l-2 = n_1 + n_2 + \dots + n_{k-1}$. Now, we use $\{P_{11}, P_{12}, \dots, P_{1n_1}, P_{21}, P_{22}, \dots, P_{2n_2}, \dots, P_{(k-1)1}, P_{(k-1)2}, \dots, P_{(k-1)n_{k-1}}, P_{l-1}, P_l\}$ to denote the l^* -container where $P_{ij} = \langle u, P'_{ij}, y^{ij}, y \rangle$ for $i \in \langle k-1 \rangle, j \in \langle n_i \rangle$ and $P_{l-1} = \langle u, P'_{l-1}, y^{l-1}, y \rangle$, $P_l = \langle u, P'_l, y^l, y \rangle$. We use z^{ij} to denote the vertex $z^{s(i, (y^{ij})_i)}$ for $i = 2, \dots, k-1$ (for example: if $y^{ij} = k2 \cdots (i-1)x(i+1) \cdots (k-1)1$, then $z^{ij} = 12 \cdots (i-1)x(i+1) \cdots k$). Thus, $(z^{ij}, v) \in E(A_{n,k})$.

For $i \in \langle k \rangle \setminus \{1, r\}$, if $n_i \neq 0$, we partite $A_{n,k}^{(k,i)}$ to $\cup_{j \in \langle n \rangle \setminus \{i\}} A_{n,k}^{(i,j)(k,i)}$. Let

$$\begin{aligned} A^i &= \{(y^{i1})^{s(1,i)}, (y^{i2})^{s(1,i)}, \dots, (y^{in_i})^{s(1,i)}\}, \\ B^i &= \{(z^{i1})^{s(k,i)}, (z^{i2})^{s(k,i)}, \dots, (z^{in_i})^{s(k,i)}\} \end{aligned}$$

By Lemma 2.5, there exists n_i disjoint paths $H_{i1}, H_{i2}, \dots, H_{in_i}$ from A^i to B^i such that

$$V\left(\bigcup_{j=1}^{n_i} H_{ij}\right) = V(A_{n,k}^{(k,i)}) \text{ and } H_{ij} = \langle (y^{ij})^{s(1,i)}, H_{ij}, (z^{ij})^{s(k,i)} \rangle \text{ for } 1 \leq j \leq n_j.$$

(a) $\{y^i \mid (y^i)_1 \neq (y)_1\} \leq 2$.

Note that $l-2 = n_1 + n_2 + \dots + n_{k-1} \geq 1$ and $n_1 = 0$. Without loss of generality, we can assume that $n_{k-1} \neq 0$. Since $l-3 \leq (n-k)(k-1)-3$, by Lemma 2.1, there exists a Hamiltonian path R of $A_{n,k}^{(k,k)} \setminus \{z^{21}, \dots, z^{2n_2}, \dots, z^{(k-1)1}, \dots, z^{(k-1)(n_{k-1}-1)}\}$ joining $z^{(k-1)n_{k-1}}$ to v . We set

$$M_{ij} = \begin{cases} \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(1,i)}, H_{ij}, (z^{ij})^{s(k,i)}, z^{ij}, v \rangle : i = 2, 3, \dots, k-2, j = 1, 2, \dots, n_i, \\ \text{or } i = k-1, 1 \leq j \leq n_{k-1}-1, \\ \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(1,i)}, H_{ij}, (z^{ij})^{s(k,i)}, z^{ij}, R, v \rangle : i = k-1, j = n_{k-1}. \end{cases}$$

Since $n-k \geq 2$, let $a \in \langle n \rangle \setminus \{\{(y)_i : 1 \leq i \leq k\} \cup \{(y^{l-1})_i : 1 \leq i \leq k\}\}, b \in \langle n \rangle \setminus \{1, 2, \dots, k, a\}$. Let $I = \langle n \rangle \setminus (\{i \mid n_i \neq 0 : 2 \leq i \leq k-1\} \cup \{k, a\})$. By Lemma 2.4, there exists a Hamiltonian path R_{l-1} of $A_{n,k}^{(k,a)}$ joining $(y^{l-1})^{s(1,a)}$ to $v^{s(k,a)}$ and a Hamiltonian path R_l of $A_{n,k}^{(k,I)}$ joining $y^{s(1,b)}$ to $v^{s(k,b)}$. We set

$$M_{l-1} = \langle u, P'_{l-1}, y^{l-1}, (y^{l-1})^{s(1,a)}, R_{l-1}, v^{s(k,a)}, v \rangle,$$

$$M_l = \langle u, P'_l, y^l, y^{s(1,b)}, R_l, v^{s(k,b)}, v \rangle.$$

Obviously, $\{M_{21}, \dots, M_{2n_2}, \dots, M_{(k-1)1}, \dots, M_{(k-1)n_{k-1}}, M_{l-1}, M_l\}$ forms an l^* -container of $A_{n,k}$. See figure 31 for illustration.

(b) $\{y^i \mid (y^i)_1 \neq (y)_1\} \geq 3$.

Note that $n_2 + \dots + n_{k-1} \leq l-3 \leq (n-k)(k-1)-3$. By Lemma 2.1, there exists a Hamiltonian path R of $A_{n,k}^{(k,k)} \setminus \{z^{21}, \dots, z^{2n_2}, \dots, z^{(k-1)1}, \dots, z^{(k-1)n_{k-1}}\}$ joining z^{11} to v . Let $X = \langle n \rangle \setminus \langle k \rangle = \{x_1, x_2, \dots, x_{n-k}\}$. Without loss of generality, we may assume that $y^{1i} = y^{s(k, x_i)}$ for $1 \leq i \leq n_1$, $y^{l-1} = y^{s(k, x_{n_1+1})}$, $y^l = y^{s(k, x_{n_1+2})}$ and $z^{11} = v^{s(1, x_{n_1+1})}$. By Lemma 2.4, there exists a Hamiltonian path R_i of $A_{n,k}^{(k, x_{i+1})}$ joining $(y^{1i})^{s(1, x_{i+1})}$ to $v^{s(k, x_{i+1})}$ for $1 \leq i \leq n_1$ and a Hamiltonian path R_{l-1} of $A_{n,k}^{(k, x_{n_1+2})}$ joining $(y^{l-1})^{s(1, x_{n_1+2})}$ to $v^{s(k, x_{n_1+2})}$. Let $I = \langle n \rangle \setminus \{\langle k \rangle \cup \{x_2, x_3, \dots, x_{n_1+2}\}\}$. By Lemma 2.4, there exists a Hamiltonian path R_l of $A_{n,k}^{(k, I)}$ joining $(y)^{s(1, x_1)}$ to $(z^{11})^{s(k, x_1)}$. We set

$$M_{1t} = \langle u, P'_{1t}, y^{1t}, (y^{1t})^{s(1, x_{t+1})}, R_t, v^{s(k, x_{t+1})}, v \rangle \text{ for } 1 \leq t \leq n_1,$$

$$M_{ij} = \langle u, P'_{ij}, y^{ij}, (y^{ij})^{s(1,i)}, H_{ij}, (z^{ij})^{s(k,i)}, z^{ij}, v \rangle \text{ for } 2 \leq i \leq k-1, 1 \leq j \leq n_i.$$

$$M_{l-1} = \langle u, P'_{l-1}, y^{l-1}, (y^{l-1})^{s(1, x_{n_1+2})}, R_{l-1}, v^{s(k, x_{n_1+2})}, v \rangle,$$

$$M_l = \langle u, y^l, y, y^{s(1, x_1)}, R_l, (z^{11})^{s(k, x_1)}, z^{11}, R, v \rangle.$$

Obviously, $\{M_{11}, \dots, M_{1n_1}, M_{21}, \dots, M_{2n_2}, \dots, M_{(k-1)1}, \dots, M_{(k-1)n_{k-1}}, M_{l-1}, M_l\}$ forms an l^* -container of $A_{n,k}$. See figure 32 for illustration. \square

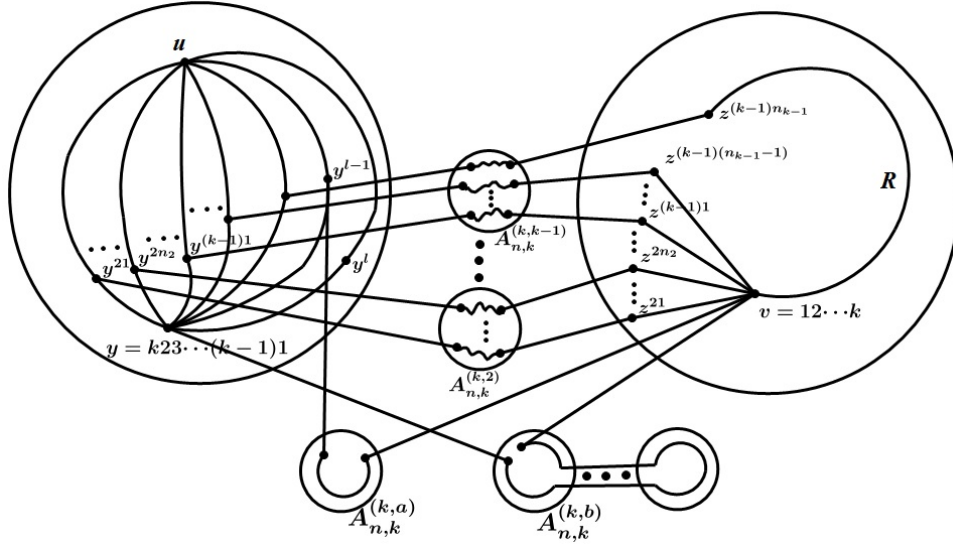


Figure 31: Illustration for subcase 2.2.2(a) of Theorem 3.2

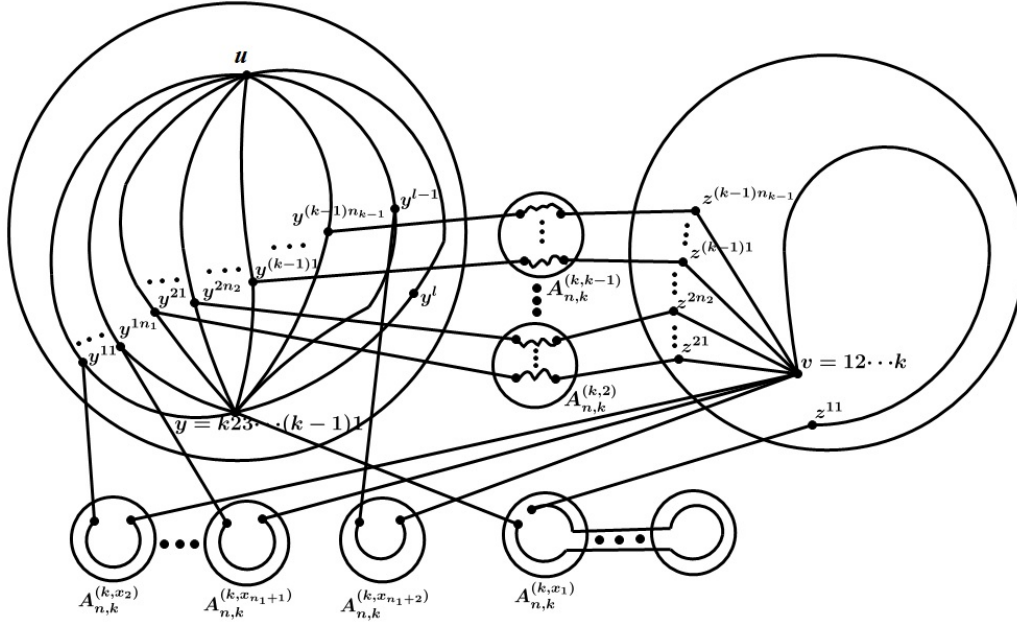


Figure 32: Illustration for subcase 2.2.2(b) of Theorem 3.2

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